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Master's thesis

Competing Information Designers

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Abstract

In modeling competition among information designers, communication is often precluded between the receiver and the designers, except when sending signals. We develop a model of competition in persuasion mechanisms, based on common agency in mechanism design, in which senders can communicate with a common receiver prior to sending their signals. In this setting, we would expect the information designers to take advantage of the possibility to communicate to acquire information on the persuasion mechanisms offered by the other senders. We show that restricting attention to direct mechanisms is without loss of generality and thus that communication about the "market" information is not needed. We also develop conditions under which an equilibrium of the game in persuasion mechanisms exists. Finally we propose an application to a certification game in which two companies compete to obtain a label.

Keywords: Information design; Bayesian persuasion; Multiple Senders; Common Receiver; Common agency; Revelation Principle.

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1 Introduction

In many economic contexts, several senders share information with a common receiver. People acquire information from several newspapers. Multiple firms design advertising campaigns addressed to the same pool of consumers. Many lobbyists try to convince one decision maker. This paper focuses on such a multi-senders common-receiver framework, with three main ingredients. First the senders can commit. So we place our-selves in the domain of information design rather than cheap talk. Our senders are information designers. Secondly there is a single common receiver, whose actions affect all information designers. In particular, the model we develop is not fit to replicate information game in which several receivers interact. The common receiver may or may not have private information. So our game is sometimes an asymmetric information game. Thirdly information designers cannot disclose credible information about the other senders. It means that two competing firms advertising their own product cannot reveal credible information about the quality of the other firm product for example. We call this framework the multi-senders common-receiver game. Our objective is to derive the tools for its analysis.

A typical example of a multi-senders common-receiver game in information design is several universities trying to persuade potential employers to hire their students. Each university can control what information to disclose on their own students, but cannot influence the perception the potential employer has about the others universities students. For example they can choose what to put on the transcripts of their students. This would typically be a public persuasion mechanism. Now suppose that there are different companies looking for workers with specific skills. Each universities can then emphasize specific qualities of its student population in its communication toward a particular recruiting firm. Commitment would be guaranteed by reputation concerns. These discriminating communication policies are private persuasion mechanisms.

There are numerous other examples that fit into our framework. In certain markets, labeled products have a clear comparative advantage. So companies supplying similar products compete to obtain a label. This is the application we develop. Political candidates disclose information to persuade voters with private bias. In a similar vein, lobbyists want to influence a politician with a private stand on a project. Firms advertising campaign are also good examples. Consumers have private preferences and firms want them to purchase their own products. In this setting, one can even add a regulator. Consider a government that would like to persuade the public to reduce the amount of drinking by commissioning an educational campaign about the health risks of alcohol, while some private companies are trying to maximize their profits by advertising alcoholic products. What is the optimal way for the government to conduct the educational campaign? Should the government address consumers differently depending on the advertising campaigns that target them? Should the government provide every consumer with the same information? This would be an interesting extension of the paper by Kolotilin et al. [18]. Two last applications could be the competition in information disclosure that the government and the central bank constantly play when talking to firms whose expectations are different; and entrepreneurs willing to convince an investor with private knowledge on his wealth to fund their project.

We adopt the framework of information design with elicitation developed by Bergemann and Morris in [7] to study competition between n senders who wants to persuade a common receiver to take a particular action when they can communicate with the common receiver. Our analysis bridges the common agency literature and the information design literature.

First we develop a framework, based on Myerson's formalism in [22], for multi-senders common-receiver games in information. Each sender can design a persuasion mechanism that reveals independent pieces of relevant information to the receiver's decision. These persuasion mechanisms are proposed prior to observing the state of the world. So we define the contracts the senders can offer to the common receiver and the competition in mechanisms game the senders play.

Once the game in mechanisms is well defined, we simplify the problem. From a game in arbitrary mechanisms, we move to a game in incentive compatible direct mechanisms. This is a trivial step in single information designer problem accomplished by the revelation principle. However when multiple information designers compete, the revelation principle is not sufficient anymore; and complications arise as shown in an example. When information designers communicate with a common receiver in a competitive environment, the receiver has information about what is happening in the market that senders do not have when they design direct contracts. If senders ask the common receiver about this "market information", i.e., what persuasion mechanisms the other senders offer, the information designers may be able to adapt their strategies to induce new equilibrium outcomes. From the common agency literature, we suspect that they could neutralize some of the effect of competition for example. We show that this does not happen in multi-senders common-receiver information design by constructing for every equilibrium in arbitrary persuasion mechanisms a robust, payoff-equivalent equilibrium in incentive compatible direct persuasion mechanisms. These are theorems 5.1 and 5.2.

Next we develop conditions under which an equilibrium of the multi-senders common-receiver game exists based on [28].

Finally we propose an application to a certification game in which two wood producing companies seek an eco-certification from a regulatory agency with a potentially private preference parameter. This multi-senders common-receiver problem is first studied in a non strategic case. Then we find the equilibrium in a competitive case and show that it is unique. In this example, competition weakly increases information disclosure.

The paper is organized as follows. Section 2 discusses the relationship with the existing literature. Section 3 sets up our basic environment. The equilibrium concept of multi-senders common-receiver game is developed in Section 4. Section 5 presents the main theoretical results of the paper. Finally Section 6 shows an application and Section 7 discusses what still need to be done. The proofs are relegated to the appendix (section 8).

2 Relations to the existing literature

Our work falls in the literature on Bayesian persuasion that followed Kamenica and Gentzkow's article [16]. They model information disclosure when the sender can commit. In particular they prove the equivalence of "statistical experiments" and Bayes-plausible distributions of posteriors, a result that we will use extensively in our analysis of the information structures. Their article has stimulated an active literature on information disclosure games in which the senders can commit to disclosure mechanisms. Public persuasion, in which the information revealed must be identical for all receiver types, has also been covered in Rayo and Segal [27].

The framework of Bayesian persuasion includes a single receiver and a single sender. A few articles are also interested in information design with multiple senders. Ostrovsky and Schwarz [23] consider a model in which schools disclose information about the ability of their students, with the objective of maximizing the students' overall placement. They also study how placements are affected by the overall distribution of abilities. Gentzkow and Kamenica in [14] and [13] look at competition in persuasion when senders can disclose any information on the state of the world. They identify a set of conditions under which competition always increase information disclosure. In particular, the information structure must be Blackwell-connected. Our paper departs from this assumption by assuming that each sender control information disclosure in one dimension of the state the world only. Li and Norman in [19] develops an example that shows that the equilibrium can be less informative when the condition of Gentzkow and Kamenica are not satisfied.

Some recent articles take a similar stand as ours in proposing a model of competition in information disclosure in which each sender controls the information structure on one dimension of the state of the world. Koessler et al. [17] prove the existence of an equilibrium in the information game played between the sender using a result of Simon and Zame (1990). Albrecht in [2] studies competition in information disclosure and the implication for political campaigns. His paper is closely linked to [4]. In the latter, Au and Kawai provide an equilibrium existence result and characterize the equilibria in an example with two senders and in a special class of symmetric games. Their model resembles a lot our application. Their existence results is also close to our theorem 5.4 although for a finite state space and a finite action space. Our paper differs from these articles in that we assume that the receiver may have private information. So in our framework, information designers compete in persuasion mechanisms and not in information structure directly. We adopt a point of view closer to common agency in mechanism design than non-cooperative game with symmetric information.

This link between information design and mechanism design has been noted by Bergemann and Morris or Kolotilin et al. too. Informational structures in multi-receiver environments are studied in Bergemann and Morris [6] and Bergemann and Morris [7]. The latter also develops the concept of information design with elicitation when the receivers have private information. Bayesian persuasion; with multiple receivers who interacts strategically in a mechanism design framework, is also considered in [30] for example. Bayesian persuasion with asymmetric information has also been studied in a few other papers. Perez-Richet [24] and Alonso and Câmara [3] look at Bayesian persuasion with a privately informed sender. Closer to our work is maybe Kolotilin et al. [18]. They study a model of Bayesian persuasion in which a receiver is privately informed. They determine the optimal private persuasion mechanism. They also establish the equivalence between public persuasion and private persuasion in their settings. Our paper pushes forward the connection between information design with elicitation and mechanism design.

In particular, it develops the relationship between common agency in mechanism design and multi-sender common-receiver games in information design with elicitation. So it draws upon the common agency literature; which has provided an important insight into how principals support collusive outcomes and how the “failure" of the revelation principle when multiple designers compete can be overcome. Epstein and Peters’s [12] constructs the universal type space that would allow the use of the revelation principal in multi-principals mechanism design. They use the work of Mertens and Zamir on the infinite beliefs regress in Bayesian game. However this type space turns out to be very complex. So Peters, in [25] and [26], and Martimort and Stole, in [20], develop tools to solve for the equilibria of the common agency problem in mechanism design. Our paper builds upon this tools to simplify the search for equilibria in the multi-senders common-receiver game.

3 A framework for competition between information designers

3.1 Preliminaries

There are n information designers, also referred in what follows as senders (males), D_i, $i = 1, \ldots, n$. There is only one common receiver: R. The common receiver R (she) has a continuous Von-Neumann Morgenstern utility function, $u_R(a, \omega, t)$, which depends on her action $a \in A$, her type $t \in T$, and the state of the world $\omega \in \Omega$. The action space A, the type space T, and the state space Ω are assumed to be compact metric spaces.

The information designers also have Von-Neumann Morgenstern continuous utility function, $v_i(a, \omega, t)$, $i = 1, \ldots, n$, where a is the action taken by the common receiver, ω is the state of the world, and t is the type of the common receiver.

In what follows, whenever a topology is needed and is not explicitly defined, assume that it is the weak* topology. Furthermore when a space of Borel probabilities on a compact metric needs to be metrized, we use the Lévy-Prohorov distance[1]. See Chapter 11 of [11] for example.

All senders and the common receiver share a commonly known prior belief about the state of the world, $\mu_0(\omega) \in \text{int}\,(\Delta(\Omega))$, where “int" means the “interior of".[2] Furthermore the receiver R may have private information. This is fully summarized by her type $t \in T$. The senders also share a common prior belief about the distribution of t on T: $t \sim F(t)$.

The senders want to maximize their expected payoffs. No transfer is allowed, but the senders can design “statistical experiments" or signal technologies to influence the common receiver’s behavior.

[1] We could equivalently use the dual-bounded Lipschitz distance.

[2] Whenever $\Delta(X)$ appears, where X is a compact metric space, it stands for the space of all Borel probabilities over X.

Hence the latter observes the outcome of the "statistical experiments" conducted by the designers and updates her beliefs using Bayes rule. Given her new beliefs on the state of the world, the receiver chooses what action to take among her available actions $a \in A$. So she alters her behavior depending on her new beliefs on the state of the world; and her new behavior may be beneficial to some senders.

The senders design "experiments" or signal technology that are the most beneficial to them. An "experiment" (π_i, S_i) designed by D_i consists of a compact metric signal space S_i and a mapping

$$\begin{aligned} \pi_i : & [0,1] \to \Omega \times S_i \\ & x \to (\pi_i^a(x), \pi_i^b(x)) \end{aligned}$$

Following [16], we define π_i to be a measurable function whose second component, $\pi_i^b(x) = s_i$, is correlated with the first, $\pi_i^a(x) = \omega$. The first component is the state of the world and the second component is the signal realization. x is uniformly distributed. The senders, D_i, $i = 1, \ldots, n$, and the common receiver R observe the realizations of all signals $\pi_i^b(x) = s_i$, $i = 1, \ldots, n$.

Conditional on a realization of signals $s = (s_1, \ldots, s_n)$, where $s_i \in S_i$ for all $i \in \{1, \ldots, n\}$, the receiver updates her beliefs on ω using Bayes rule:

$$\mu_s(\omega) = \frac{\pi(s \mid \omega)\mu_0(\omega)}{\int_\Omega \pi(s \mid \omega)\mu_0(\omega)}$$

where $\pi(s \mid \omega)$ is the joint distribution on s conditional on the state of the world being ω. μ_s is the posterior belief on ω conditional on signal s being observed.

The receiver is indeed able to compute a posterior probability on ω when $\forall i$ S_i is a compact metric space (therefore complete and separable). Hence when it is the case, there exists a regular conditional probability as noted in Kamenica and Gentzkow [16]'s online appendix.

Hence for any measurable space $\mathcal{R}$, any prior distribution $P \in \Delta(\Omega \times \mathcal{R})$, and any integrable function $y : \Omega \times \mathcal{R} \to \mathbb{R}$, there exists a $\mathcal{R}$-measurable function $\mathbb{E}[y \mid r, P]$ such that

$$\int_D E[y \mid r, P]\, dP = \int_D y dP$$

for all $D \subset \mathcal{R}$ (see [10], p. 393). let $\mathbf{1}_O$ the characteristic function of any measurable subset $O \subset \Omega$. Then $\mathbb{E}[\mathbf{1}_O \mid r, P]$ defines a conditional probability measure in $\Delta(\Omega)$, interpreted as the posterior probability for O conditional on r and the prior $P \in \Delta(\Omega \times \mathcal{R})$.

Using the posterior μ_s, the receiver R of type t decides what action $a \in A$ to take to maximize $\mathbb{E}_{\mu_s}[u_R(a,\omega,t)]$ [3]. However, μ depends on all signals being played. So the senders D_i, $i = 1,\ldots,n$, compete to influence the common receiver's action in the most favorable way to them. They play the strategy that maximizes their expected payoffs given the initial prior, the strategies of the other senders, and the action optimally chosen by R for any induced beliefs. We are interested in the Bayesian Nash equilibria of the multi-senders single-receiver game defined above.

3.2 Competition in mechanisms

The problem of sender D_i is to determine the strategy that maximizes his expected utility. We authorize communication between the senders and the receiver. Our framework corresponds to information design with elicitation (Bergemann and Morris, [7]) or, as put by Kolotilin et al. [18], private persuasion.

Each sender D_i can design a test (a communication mechanism) that ask the receiver to report her private information and picks a signal technology conditional on her report and the realized state of the world, in order to influence the decision made by the receiver. As noted by Bergemann and Morris [7], the case of private persuasion is closely related to mechanism design; and we adopt this perspective. The senders compete in mechanisms.

We will use Myerson [22]'s formalism to model competition between the senders. Begin with an assumption.

Assumption 1. *The common receiver is bound by the mechanisms offered by the information designers*[4].

The mechanisms offered to the receiver determine her possible actions. Theses mechanisms are defined below and will be referred in what follows as persuasion mechanisms. They are composed of a communication space, a set of simple actions, and a contract, i.e., a mapping from the communication space into the set of simple actions.

Set of simple actions A persuasion mechanism maps a message space to a set of simple actions that can be taken by principals. Contrary to mechanism designs, here, the set of simple actions does

[3] $\mathbb{E}_\mu$ corresponds to the expectation operator for a random variable distributed according to μ, i.e., $\mathbb{E}_\mu \cdot \equiv \int_{\text{supp } \mu} \cdot \, d\mu$.

[4] In particular, participation decision, if considered, is incorporated in the receiver's set of actions.

not include transfer. Instead it is the set of all "statistical experiments" that may be designed by a sender, already defined above. So the set of simple actions available to information designer D_i is $\mathcal{E}_i = \{(\pi_i, S_i)\}$, where (π_i, S_i) is a signal technology or "statistical experiment". S_i is a compact metric signal space[5] and $\pi_i : [0,1] \to \Omega \times S_i$ is a measurable function.

Each simple action generates a signal when it maps a uniform random variable x to $\pi(x) = (\pi_i^a(x), \pi_i^b(x))$, where $\pi_i^a(x) = \omega$ describes the state of the world and $\pi_i^b(x) = s_i$ is the signal sent by D_i.

To further characterize $\mathcal{E}_i$, we make the following assumption.

Assumption 2. *The state of the world is a n-dimensional vector $\omega = (\omega_1, \dots, \omega_n) \in \Omega$. For all $i \neq j$, ω_i and ω_j are independent random variables. They are independent of the receiver's type t too. $\forall i$, $\omega_i \sim \mu_i$ and $\omega \sim \mu = \prod_{i=1}^n \mu_i$. Information designer D_i can only display credible information about ω_i.*

The above assumption implies that all senders are experts in one dimension of the state of the world only. For example, if the information designers are firms, it means that they can only display information about their own product. It departs from Gentzkow and Kamenica model of competition in persuasion [14] since it implies that the information set we consider is not Blackwell-connected[6].

Then the set of simple actions available to sender D_i is $\mathcal{E}_i = \{(\pi_i, S_i)\}$ such that π_i is a measurable mapping from $[0,1]$ into $\Omega_i \times S_i$. Furthermore, from Kamenica and Gentzkow [16]'s online appendix, there exists a one-to-one mapping between the set of all "statistical experiments" and the set of Bayes-plausible distributions of posteriors. A distribution of posteriors, denoted by τ_i, is an element of the set of Borel probabilities on the compact metric space $\Delta(\Omega_i)$: $\tau_i \in \Delta(\Delta(\Omega_i))$, and a distribution $\tau_i \in \Delta(\Delta(\Omega_i))$ is Bayes-plausible if

$$\int_{\Delta(\Omega_i)} \mu_i \, d\tau_i(\mu_i) = \mu_0^i$$

where μ_0^i is the marginal distribution on ω_i derived from the prior distribution μ_0 on ω.

So for any Bayes-plausible τ_i, there exists a signal technology (π_i, S_i) that induces it, and any signal technology (π_i, S_i) induces a posterior that is Bayes-plausible, given that x is uniformly distributed. The set of simple actions that senders can then be replaced by the set of all Bayes-plausible

[5] The signal space is unique here without loss of generality. If some signals are only needed in some cases, then their probability is set to zero the rest of the time.

[6] In this case, the equilibria of the game are not characterized by the maximal display of information, and must be determined in a different way.

distributions τ_i on $\Delta(\mu_i)$. Rewrite

$$\mathcal{E}_i = \left\{ \tau_i \in \Delta(\Delta(\Omega_i)) \text{ such that } \int_{\Delta(\Omega_i)} \mu_i \, d\tau_i(\mu_i) = \mu_0^i \right\}$$

Note also that allowing for randomization over simple actions does not change the set of simple actions. Hence

Lemma 3.1. *For any randomization over simple actions, there exists a non-random simple action that induces the same distribution on posterior beliefs.*

Proof. Let $\delta_i \in \Delta(\mathcal{E}_i)$. Consider $\tau_i \in \mathcal{E}_i$ such that $\text{supp } \tau_i = \underset{\tau' \in \text{supp } \delta_i}{\cup} \text{supp } \tau'$ and $\forall \mu \in \text{supp } \tau_i$

$$\tau_i(\mu) = \int_{\mathcal{E}_i} \tau'(\mu) \, d\delta_i(\tau')$$

Then $\tau_i \in \mathcal{E}_i$ induces the same ex-ante distribution of posteriors on $\Delta(\Omega_i)$. □

The set of simple actions is the set of actions that senders can take without communicating with the common receiver. It will be the image set of the contracts.

Communication space Each information designer D_i chooses an arbitrary measurable communication space $\mathcal{M}_i$. The communication space is the set of messages the receiver can send to D_i. In particular, the communication space chosen by sender D_i is not restricted to T. It may be large enough to communicate the mechanisms chosen by the other senders[7].

Denote $\mathcal{M} = \prod_{i=1}^n \mathcal{M}_i$ the communication space offered to the common receiver.

Contract Finally, a persuasion mechanism also includes a contract, i.e., a measurable mapping from the communication space $\mathcal{M}_i$ into the set of simple actions $\mathcal{E}_i$.[8] We denote this mapping by

$$\gamma_i : \mathcal{M}_i \to \mathcal{E}_i$$

The designers can only contract on the messages that are directly addressed to them and they have commitment power. If D_i offers a mapping γ_i that associates the simple action a to the message m, he cannot play another simple action $a' \neq a$ when the receiver sends message m to him.

[7]Such communication space exists, as the infinite regress generated by mechanisms depending on mechanisms depending on mechanisms. . . was proved to converge to an "universal communication space" by Epstein and Peters [12].

[8]Although considering only measurable mechanims may look like a restriction, it is not by lemma 3.1.

Persuasion mechanisms We are now equipped to give a rigorous definition of a persuasion mechanism.

Definition 1. *A persuasion mechanism for information designer D_i is a tuple $(\mathcal{M}_i, \gamma_i)$ where $\mathcal{M}_i$ is a communication space and γ_i is a contract.*

We will index any persuasion mechanism by its contract, i.e., $\gamma_i = (\mathcal{M}_i, \gamma_i)$ with a slight abuse of notation. So a persuasion mechanism associates to a receiver report a distribution on posterior beliefs on ω_i given the unobserved state of the world.

We say that a persuasion mechanism is private if $|\mathcal{M}_i| \geq 2$. A private persuasion mechanism can discriminate in function of the report of the common receiver. On the contrary, a persuasion mechanism is said to be public if $|\mathcal{M}_i| \leq 1$. The two setting can be compared as in [18]. When a mechanism is public, the sender chooses a signal technology independently of the report of the common receiver. In particular, in a public persuasion mechanism framework, any type of receiver is informed identically. Observe that all public persuasion mechanisms can be replicated by a private persuasion mechanism such that

$$\gamma_i(m_i) = e_i \in \mathcal{E}_i, \quad \forall m \in \mathcal{M}_i$$

Finally let Γ_i be the set of all feasible mechanisms available to information designer D_i. Γ_i is arbitrary since $\mathcal{M}_i$ is arbitrary. Define also $\Gamma = \prod_{i=1}^{n} \Gamma_i$, the set of feasible array of mechanisms $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$ in the competition in mechanisms game played by the information senders.

Competition in persuasion mechanism We are interested in the competition in persuasion mechanisms between the senders and in the display of information that results from this competition. We want to determine the strategic behaviors of the information designers. Note that we allow the senders to play randomization on mechanisms; so the strategies of sender D_i lies in $\Delta(\Gamma_i)$. Finally we assume that

Assumption 3. *Senders can design credible tests costlessly. These tests are payoff irrelevant to the common receiver conditional on the signal realization. They are also payoff irrelevant to all information designers.*

Therefore the only goal of the senders when picking a mechanism to play is to influence the action of the unique receiver.

3.3 Receiver's behavior

The common receiver R faces the following decision problem. Given the persuasion mechanisms chosen by the designers, she chooses what message to send to each of them, i.e., she chooses an array of message $m = (m_1, \ldots, m_n) \in \mathcal{M}$. Then, given messages, the information designers displays some information on ω associated with the contracts $(\gamma_1(m_1), \ldots, \gamma_n(m_n))$. It induces posterior beliefs μ_s on ω drawn from the joint distribution of posterior beliefs associated with the contracts $(\gamma_1(m_1), \ldots, \gamma_n(m_n))$. Then, given her new beliefs μ_s, the receiver plays her optimal action.

So receiver R's behavior depends on her utility function and on the mechanisms offered by the senders. Following Peters [25], we define a communication strategy $\tilde{m}$ to be a measurable mapping[9]

$$\tilde{m} : T \times \Gamma \to \Delta(\mathcal{M})$$

It describes the probability distribution on the whole set of messages that the receiver will send to the information designers once she has observed all mechanisms.

Similarly we define a decision strategy as a measurable mapping

$$\tilde{a} : T \times \Delta(\Omega) \to \Delta(A)$$

It describes the probability distribution on actions that the receiver chooses. Remark that this decision strategy only depends on the receiver's type and the signal realizations. In particular, it does not depend on the offered mechanisms (that are payoff irrelevant).

The receiver has no commitment power. Since the common receiver's payoff does not depend on the mechanism played directly, she cannot punish the senders for playing a mechanism rather than another conditional on the signal realization (i.e., induced beliefs). She is maximizing her expected payoff given the signal realization and deviating to punish a sender constitutes a non-credible threat.

Then two different arrays of mechanisms γ and γ' that produce the same signal realization, therefore the same posterior beliefs, induce the same distribution on actions. Furthermore, all mechanisms that have the same image space, but may have different communication spaces, produce the same interim expected distribution of actions.

Together a communication strategy and a decision strategy forms a continuation strategy: $c =$

[9] We adopt the convention that $\tilde{\cdot}$ refers to a distribution.

$(\tilde{m}, \tilde{a})$.

3.4 Timing of the game

Subsections 3.1, 3.2, and 3.3 describes the framework of our multi-senders common-receiver persuasion game. This subsection summarizes what has been developed above and insists on the timing of the game for clarity. The persuasion game proceeds as follows

- At time $t = 0$, nature draws $x \sim \mathcal{U}[0, 1]$. No player observe the draw. This determines the state of the world.
- At time $t = 1$, all information designers D_i, $i = 1, \ldots, n$, simultaneously propose a persuasion mechanism. They may randomize over several mechanisms.
- At time $t = 2$, the common receiver R optimally chooses her communication strategy $\tilde{m}$.
- At time $t = 3$, signals are generated and the receiver updates her beliefs on ω using Bayes law.
- At time $t = 4$, the common receiver R optimally chooses her decision strategy $\tilde{a}$ given her new beliefs μ_s.

4 Equilibrium

An equilibrium for the multi-senders common-receiver game is an array $\{\delta_1, \ldots, \delta_n, c\}$, where δ_i, $i = 1, \ldots, n$, is a randomization on Γ_i and c is a continuation strategy, such that no player has a profitable deviation in expected utility. We are interested in the Bayesian Nash equilibria of the game.

To characterize more precisely the equilibria, we first define what a continuation equilibrium is.

4.1 Continuation equilibrium

The receiver's problem is a two-stages problem. First the receiver chooses optimally what communication strategy to adopt, i.e., what messages to send to all designers, given the mechanisms she is offered. Since we assumed that the senders are able to commit, it induces an array of signals $(s_1 = \gamma_1(m_1)^b(x), \ldots, s_n = \gamma_n(m_n)^b(x))$, where $\gamma_i(m_i)$ is the signal technology associated with mechanism γ_i. $\gamma_i(m_i)$ is a measurable function from $[0, 1]$ into $\Omega \times S_i$ and $\gamma_i^b(x)$ is the signal realization. This signals realizations are publicly observed.

From subsection 3.2, we know that this array of signals generates posterior beliefs on ω: the receiver updates her belief on ω. Then she optimally choose what action a to undertake given her new beliefs μ_s.

The continuation strategy played in equilibrium by the receiver, $c = (\tilde{m}, \tilde{a})$ is determined by solving the receiver's problem using backward induction.

4.1.1 Equilibrium decision strategy

In the second stage of her problem, the common receiver has beliefs μ_s. From her posterior beliefs μ_s, she determines her optimal decision strategy in $\Delta(A)$ to maximize

$$\mathbb{E}_{\mu_s}\left[u_R(a,\omega,t)\right] = \int_\Omega \int_A u_R(a,\omega,t)\, d\tilde{a}(t,\mu_s)\, d\mu_s(\omega)$$

This strategy is independent of ω as the state of the world is not observable. This implies that the receiver should put all the mass of $\tilde{a}(t,\mu) \in \Delta(A)$ at the maximizer of her expected utility $\mathbb{E}_{\mu_s}\left[u_R(a,\omega,t)\right]$ since, by Fubini,

$$\int_\Omega \int_A u_R(a,\omega,t)\, da(t,\mu_s)\, d\mu_s(\omega) = \int_A \int_\Omega u_R(a,\omega,t)\, d\mu_s(\omega)\, da(t,\mu_s)$$

Therefore any mixed decision strategy that is part of a continuation equilibrium assigns zero probability to every action $a \in A \setminus a^*(\mu_s,t)$ where $a^*(\mu_s,t) = \arg\max_{a\in A} \int_\Omega u_R(a,\omega,t)\, d\mu_s(\omega)$. We formalize the above heuristic argument below.

Proposition 4.1. *Let $c = (\tilde{m},\tilde{a})$ be a continuation equilibrium. For any beliefs μ_s, any type $t \in T$, and any Borel set $\mathcal{A} \subset A$,*

$$\tilde{a}(t,\mu_s)(\mathcal{A}) = \tilde{a}(t,\mu_s)\left(\mathcal{A} \cap a^*(\mu_s,t)\right)$$

where $\tilde{a}(t,\mu_s)(\cdot)$ is the probability measure associated with the continuation equilibrium c. In particular, if $\mathcal{A} \cap a^*(\mu_s,t) = \emptyset$,

$$\tilde{a}(t,\mu_s)(\mathcal{A}) = 0$$

Proof. The second part is an obvious consequence of the first part. So we only have to show the first part. We do so by contradiction. Suppose that $\tilde{a}$ is a decision strategy that is part of a continuation equilibrium, and there exists $\mathcal{A} \subset A$ such that $\tilde{a}(\mu_s,t)(\mathcal{A} \cap a^*(\mu_s,t)) \neq \tilde{a}(\mu_s,t)(\mathcal{A})$. Trivially $\mathcal{A} \not\subset$

$a^*(\mu_s, t)$.

Then $\tilde{a}(mu_s, t)(\mathcal{A} \cap a^*(\mu_s, t)) < \tilde{a}(mu_s, t)(\mathcal{A})$ since $\mathcal{A} \cap a^*(\mu_s, t) \subset \mathcal{A}$. But $\mathcal{A} = \{\mathcal{A} \setminus (\mathcal{A} \cap a^*(\mu_s, t))\} \cup \{\mathcal{A} \cap a^*(\mu_s, t)\}$.

Then $\tilde{a}(\mu_s, t)(\mathcal{A} \setminus (\mathcal{A} \cap a^*(\mu_s, t))) > 0$, and we distinguish two cases.

First, suppose that $\mathcal{A} \cap a^*(\mu_s, t) = \emptyset$. Then $\tilde{a}(\mu_s, t)(\mathcal{A}) > 0$. Define $\epsilon > 0$ such that $\tilde{a}(\mu_s, t)(\mathcal{A}) - \epsilon > 0$ and consider the following strategy

$$\tilde{a}'(\mu_s, t)(\mathcal{A}') = \begin{cases} \tilde{a}(\mu_s, t)(\mathcal{A}') - \epsilon \tilde{a}(\mu_s, t)(\mathcal{A}' \cap \mathcal{A}) \text{ if } \mathcal{A} \cap \mathcal{A}' \neq \emptyset \text{ and } \bar{a} \notin \mathcal{A}' \\ \tilde{a}(\mu_s, t)(\mathcal{A}') + \epsilon(\tilde{a}(\mu_s, t)(\mathcal{A}') - \tilde{a}(\mu_s, t)(\mathcal{A}' \cap \mathcal{A})) \text{ if } \bar{a} \in \mathcal{A}' \text{ and } \mathcal{A} \cap \mathcal{A}' \neq \emptyset \\ \tilde{a}(\mu_s, t)(\mathcal{A}') \text{ otherwise} \end{cases}$$

where $\bar{a} \in a^*(\mu_s, t)$. $\tilde{a}'(\mu_s, t)(\cdot)$ is indeed a probability measure over A, so it is a decision feasible strategy. Furthermore

$$\begin{aligned} \int_A \int_\Omega u_R(a, \omega, t)\, d\mu_s(\omega)\, d\tilde{a}'(\mu_s, t)(a) - \int_A \int_\Omega u_R(a, \omega, t)\, d\mu_s(\omega)\, d\tilde{a}(\mu_s, t)(a) \\ \geq \epsilon \int_\Omega u_R(\bar{a}, \omega, t)\, d\mu_s(\omega) - \epsilon \sup_{a \in \mathcal{A}} \int_\Omega u_R(a, \omega, t)\, d\mu_s(\omega) \\ > 0 \end{aligned}$$

since $\mathcal{A} \cap a^*(\mu_s, t) = \emptyset$. Therefore we reach a contradiction: there is a profitable deviation so $\tilde{a}$ cannot be part of a continuation strategy.

Secondly, suppose that $\mathcal{A} \cap a^*(\mu_s, t) \neq \emptyset$. Then $\mathcal{A} \setminus a^*(\mu_s, t) \neq \emptyset$, $\{\mathcal{A} \setminus a^*(\mu_s, t)\} \cap a^*(\mu_s, t) = \emptyset$ and $\tilde{a}(\mu_s, t)(\mathcal{A} \setminus a^*(\mu_s, t)) > 0$. Thus the same reasoning as above yields a contradiction. □

The above proposition is really that the support of any mixed decision strategy is included in the set of maximizers of the expected utility of the receiver, which should be obvious. Any decision strategy played in a continuation equilibrium belongs to $\Delta(a^*(\mu_s, t))$ for some μ_s induced by the offered mechanism and the communication strategy played in this continuation equilibrium.

Let $\tilde{a}^*(\mu_s, t)$ denote the set of optimal decision strategies for the common receiver of type $t \in T$ given her beliefs are $\mu_s \in \Delta(\Omega)$:

$$\tilde{a}^*(\mu_s, t) = \arg \max_{\tilde{a} \in \Delta(A)} \int_A \int_\Omega u_R(a, \omega, t)\, d\mu_s(\omega) d\tilde{a}(a)$$

From the above proposition, we know that $\tilde{a}^*(\mu_s, t) \subset \Delta(a^*(\mu_s, t))$ for all $t \in T$. Furthermore since $\int_\Omega u_R(a, \omega, t)\, d\mu_s(\omega)$ is continuous in a, by the linearity of the integral, and in μ_s, by Lebesgue's dominated convergence theorem, and since A is compact, Berge's maximum theorem yields that $a^*(\mu, t)$ is a an upper hemicontinuous, non-empty valued, compact valued, correspondence from $\Delta(\Omega)$ into A. Therefore, as a direct consequence of Helly's selection theorem, the set $\tilde{a}^*(\mu, t)$ is compact too, since there exists a homeomorphism (the cumulative distribution functions) between the space of Borel probabilities on a compact set and the set of non-decreasing right-continuous functions on this compact set into $[0, 1]$. Furthermore since $a^*(\mu, t)$ is non-empty, so is $\tilde{a}^*(\mu, t)$. Thus $\tilde{a}^*(\mu, t)$ is a non-empty compact subset of $\Delta(A)$[10].

Indifference problem However, the set $\tilde{a}^*(\mu, t)$ is not necessarily a singleton; and in our framework, there are n different senders. So we cannot rule out the potential problem of the existence of multiple optimal decision strategies for the receiver by picking the one preferred by the senders since different senders may have different preferences over the action played by the receiver. Therefore the concept of senders-preferred subgame perfect equilibrium of Kamenica and Gentzkow [16] is not directly applicable. Instead, we will be breaking the potential tie between receiver's multiple maximizing decision strategies by defining a new simple bargaining subgame.

When the receiver is indifferent between several decision strategies given her beliefs μ_s, she picks the one that maximizes the surplus function V_2:

$$V_2(\tilde{a}, t) = \sum_{i=1}^{n} \mathbb{E}_{\mu_s} \mathbb{E}_{\tilde{a}} v_i(a, \omega, t)$$

This function can be thought to represent a bargaining game where the bargaining power of all information designer D_i is identical[11].

Define then $\tilde{a}^c(\mu_s, t) \in \tilde{a}^*(\mu_s, t)$, the optimal continuation decision strategy, as the distribution in $\tilde{a}^*(\mu, t)$ on receiver's possible action A[12] that is solution to the bargaining subgame. $\tilde{a}^c(\mu_s, t)$ always exists since $V_2(\tilde{a}, t)$ is continuous in $\tilde{a}$ by Lebesgue's dominated convergence theorem and we showed

[10]A maybe more direct proof would use first Helly's selection theorem to establish the compactness of $\Delta(A)$, and then Berge's maximum theorem to derive the compactness and non-emptyness of $\tilde{a}^*(\mu, t)$, where the continuity in $\tilde{a}$ comes from Lebesgue's dominated convergence theorem. Note also that the compactness of $\Delta(A)$ could be established as a consequence of Riesz representation theorem and Alaoglu's theorem. See for example proposition 5.3 in [31], and note that the convergence in the Lévy–Prokhorov metric is the same as weak convergence when the underlying space is separable.

[11]If the set of maximizer is still not a singleton, assume that the receiver picks any remaining continuation equilibrium.

[12]$\tilde{a}^c(\mu, t)$ is extended to A by assigning probability zero to all $a \notin a^*(\mu_s, t)$.

above that $\tilde{a}^*(\mu_s, t)$ is non-empty and compact. So $\tilde{a}^c(\mu_s, t)$ exists as a corollary of Heine's theorem.

The possibility of using a randomization on actions comes from the bargaining subgame. Hence there is a priori no reason for the receiver to randomize on actions, but a mixed strategy can be the outcome of the bargaining subgame.

At this stage of the game, and given the signal realization s induces the beliefs μ_s on ω, the interim expected payoffs of the common receiver R of type $t \in T$ is

$$\hat{u}_R(\mu_s, t) = \int_\Omega \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu_s, t)(a)\, d\mu_s(\omega)$$

4.1.2 Equilibrium communication strategy

In the first stage of her problem, the common receiver R is offered an array of mechanisms $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$ and therefore a communication space $\mathcal{M} = \prod_{i=1}^n \mathcal{M}_i$. She determines her optimal communication strategy, i.e., what messages to send to the information designers, to maximize her expected continuation utility in the second stage. So for any array of mechanisms $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, the receiver chooses her communication strategy $\tilde{m} \in \Delta(\mathcal{M})$ to maximize

$$\begin{aligned}
&\max_{\tilde{m} \in \Delta(\mathcal{M})} \Big\{ \mathbb{E}_{\tilde{m}} \mathbb{E}_{\gamma(m)} \mathbb{E}_\mu \mathbb{E}_{\tilde{a}^c(\mu,t)} u_R(a, \omega, t) \\
&= \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \hat{u}_R(\mu, t)\, d\gamma_n(m_n)(\mu_n) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}(m_1, \ldots, m_n) \Big\}
\end{aligned}$$

She thus chooses a communication strategy from the set

$$\tilde{m}^*(\gamma, t) = \arg\max_{\tilde{m} \in \Delta(\mathcal{M})} \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \hat{u}_R(\mu, t)\, d\gamma_n(m_n)(\mu_n) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}(m_1, \ldots, m_n)$$

Indifference problem Again, as in the second stage, the set $\tilde{m}^*(\gamma, t)$ is not necessarily a singleton and in our framework, there are n different senders. So we cannot rule out the potential problem of the existence of multiple optimal communication strategies for the receiver by picking the one preferred by the senders since different senders may have different preferences over the messages communicated by the receiver. Therefore the concept of sender-preferred subgame perfect equilibrium of Kamenica and Gentzkow [16] is again no longer applicable. Instead, we will be breaking the tie between receiver's multiple decision strategies by defining a new simple bargaining subgame similar to the tie breaking subgame for decision strategies.

First when the receiver is indifferent between several communication strategies given the array of mechanisms offered, she always tell the truth, if it constitutes an optimal communication strategy. This will later guarantee that we will be able to restrict attention to truthful persuasion mechanisms.

If truthful reporting is not optimal and the receiver is indifferent between several communication strategies, given the array of mechanisms offered, she picks the one that maximizes the surplus function V_1:

$$V_1(\tilde{m}) = \sum_{i=1}^{n} \mathbb{E}_{\tilde{m}} \bar{v}_i(\gamma(m), t)$$

where

$$\bar{v}_i(\gamma(m), t) = \int_{\Delta(\Omega_1)} \dots \int_{\Delta(\Omega_1)} \int_{\Omega} \int_{A} v(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(m_n)(\mu_n) \dots d\gamma_1(m_1)(\mu_1)$$

and $\tilde{a}^c(\mu, t)$ is the continuation equilibrium decision strategy played in the second stage given that beliefs on ω is μ. This function can be thought to represent a bargaining game where the bargaining power of every information designer D_i is identical[13].

Define (when it exists) $\tilde{m}^c(\gamma, t) \in \tilde{m}^*(\gamma, t)$, the optimal continuation communication strategy. It is the distribution on receiver's possible messages $\mathcal{M}$ in $\tilde{m}^*(\gamma, t)$ that is solution to the bargaining subgame.[14]

$\tilde{m}^c(\gamma, t)$ is not guaranteed to exist since the communication spaces offered by the information designers can be arbitrary (and in particular do not need to be compact). Thus, in what follows, we will assume that such a communication strategy exists.[15]

Again there is a priori no reason for the receiver to use a randomized communication strategy, she could pick any pure strategy that maximizes her expected utility. However, a mixed communication strategy can be the outcome of the bargaining game selecting the continuation equilibrium played. That's why we consider this possibility. However when truthful reporting is optimal, the receiver always plays it. So she plays a pure communication strategy.

[13] As in the second stage, if the set of maximizer is still not a singleton, assume that the receiver picks any remaining communication strategy

[14] $\tilde{m}^c(\gamma, t)$ is extended to $\mathcal{M}$ by assigning probability zero to all $m \notin m^*(\gamma, t)$.

[15] Note that by an argument similar to the one for the existence of a decision strategy, a continuation communication strategy does exists whenever $\mathcal{M}$ is compact.

Then, for any array of mechanisms offered by the senders, the communication strategy chosen by the agent induces a distribution over posterior beliefs given by

$$\tau^{\gamma}(\mu) = \int_{\mathcal{M}} \gamma_1(m_1)(\mu_1) \dots \gamma_n(m_n)(\mu_n) d\tilde{m}^c(m)$$

where $\tilde{m}^c$ solves the above maximization problem. Given this distribution over posterior beliefs, the expected payoff of the receiver is

$$u_R^c(\gamma, t) = \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \dots \int_{\Delta(\Omega_n)} \hat{u}_R(\mu, t)\, d\gamma_n(m_n)(\mu_n) \dots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c(m_1, \dots, m_n)$$

4.1.3 Equilibrium continuation strategy

Given a persuasion mechanism, the common receiver chooses her continuation strategy. The pair $c(\gamma, t) = (\tilde{m}^c(\gamma, t), a^c(\mu, t))$ constitutes a continuation equilibrium if and only if $\forall \gamma \in \Gamma$ and $\forall t \in T$

$$\tilde{m}^c(\gamma, t) \in \tilde{m}^*(\gamma, t), \tag{1}$$

$\forall a.e.\, \mu \in \operatorname{supp} \tau^{\gamma}(\mu)$

$$\tilde{a}^c(\mu, t) \in \tilde{a}^*(\mu, t), \tag{2}$$

where τ^{γ} is the distribution of posteriors induced by $\tilde{m}^c(\gamma, t)$, and $\tilde{m}^c(\gamma, t)$ and $\tilde{a}^c(\mu, t)$ are solutions of the bargaining subgames.

Condition (1) tells us that the communication strategy the receiver uses maximizes her expected utility given her decision strategy. Condition (2) tells us that the receiver maximizes her expected utility conditional on the signals' realizations she observes.

4.2 The information designers' game in mechanisms

The continuation strategy of the common receiver (when it exists) defines a normal form game for the senders in which the action space is Γ. Hence, given an array of mechanisms $\gamma \in \Gamma$ is offered, the receiver plays a continuation equilibrium $c(\gamma, t) = (\tilde{m}^c(\gamma, t), \tilde{a}^c(\mu, t))$. This induces a distribution on beliefs $\tau^{\gamma}(\mu, t)$ and therefore a distribution on actions $\tilde{a}^{\gamma}(t) = \tau^{\gamma}(\mu)\tilde{a}^c(\mu, t)$ conditional on the type

$t \in T$ of the receiver. Then the expected payoff of any information designer D_i is given by

$$\bar{v}_i(\gamma) = \int_T \int_{\Delta(\Omega)} \int_\Omega \int_A v(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\tau^\gamma(\mu, t)\, dF(t)$$

Our equilibrium concept for the senders game is the Bayesian Nash equilibrium concept. An equilibrium in the senders game relative to a continuation strategy and a set of feasible mechanisms $\Gamma = \prod_{i=1}^n \Gamma_i$ is then defined as an array of randomizations $\{\delta_1, \ldots, \delta_n\}$ such that no sender D_i, $i \in 1, \ldots, n$, has a unilateral profitable deviation in expected payoff. Formally,

Definition 2. $\{\delta_1, \ldots, \delta_n\} \in \prod_{i=1}^n \Delta(\Gamma_i)$ *is an equilibrium of the senders game relative to* Γ *if*

$$\forall i \in \{1, \ldots, n\},\, \forall \delta'_i \in \Delta(\Gamma_i), \quad \mathbb{E}_{(\delta_i, \delta_{-i})} \bar{v}_i(\gamma_i, \gamma_{-i}) \geq \mathbb{E}_{(\delta'_i, \delta_{-i})} \bar{v}_i(\gamma'_i, \gamma_{-i})$$

Any equilibrium is defined relatively to a set of feasible mechanisms. Two set of mechanisms are a priori of particular interest to us: the set of public persuasion mechanisms and the set of universal persuasion mechanisms.

- An array of randomizations $\{\delta_1, \ldots, \delta_n\}$ is an equilibrium in public persuasion mechanism if there is no deviation in public persuasion mechanisms; that is an equilibrium when no significant communication between the senders and the receiver is allowed. Recall that a persuasion mechanism is said to be public when $|\mathcal{M}_i| \leq 1$. Then a public persuasion mechanism equilibrium is an equilibrium in take it or leave it offer in simple actions. Formally an array $\{\delta_1, \ldots, \delta_n\} \in \prod_{i=1}^n \Delta(\mathcal{E}_i)$ is an equilibrium in public persuasion mechanism if and only if

  $$\forall i \in \{1, \ldots, n\},\, \forall \delta'_i \in \Delta(\mathcal{E}_i), \quad \mathbb{E}_{(\delta_i, \delta_{-i})} \bar{v}_i(e_i, e_{-i}) \geq \mathbb{E}_{(\delta'_i, \delta_{-i})} \bar{v}_i(e'_i, e_{-i})$$

  From lemma 3.1, there is no reason to use mixed strategies in this case: any equilibrium in mixed strategies on simple actions can be replicated by an equilibrium in pure strategies.

- Let Γ^u be the universal set of mechanisms, which is defined by the universal type space constructed by Epstein and Peters in [12]. An array of randomizations $\{\delta_1, \ldots, \delta_n\} \in \Gamma^u$ is an equilibrium relative to all possible feasible mechanisms if there is no deviation in persuasion mechanism. Hence the set of equilibria relative to Γ^u is universal and robust: it includes all equilibria information designers can achieve when they can design arbitrary persuasion mechanisms.

An equilibrium of the multi-senders common-receiver game relative to a set of feasible mechanism Γ is thus an array $(\delta_1, \ldots, \delta_n, c) \in \prod_{i=1}^n \Delta(\Gamma_i) \times \Delta(\mathcal{M}) \times \Delta(A)$, where $(\delta_1, \ldots, \delta_n)$ is an Bayesian Nash equilibrium of the senders normal form game defined by c and c is the continuation equilibrium associated with the arrays of mechanisms offered.

4.3 Multiple information designers and the revelation principle

The above characterization of the game and its equilibria voluntarily remains very general. In particular, the set of mechanisms that can be played by the information designers is not restricted to the set of incentive compatible direct mechanisms, as we are used to. This is because the revelation principle may "fails" when several senders compete, or, to be more precise, because the type $t \in T$ of the common receiver is no longer sufficient to summarize all the private information she has.

As well known in the common agency literature, when several principals contract with a single common agent, the revelation principle must be used with caution. Hence truthful incentive compatible direct mechanisms are much harder to construct since the state of the world is not perfectly described by the type of the agent anymore. In particular, the agent has "market information". She observes the mechanisms offered by all principals. Since, in theory, principals are not constrained on the mechanisms they propose, it is realistic to imagine that they may condition their mechanisms on the agent's type but also on the other mechanisms she is offered, leading to a possible infinite regress. Such infinite regress is in its nature similar to the possible infinite regress over beliefs in a Bayesian game.

However contrary to the regress over belief that can be handle easily with the common prior assumption, there is no trick to easily fix the problem in multiple-principals agent contracting. In particular, the revelation principle looses much of its utility as the universal type space[16] under which it remains valid is too big to be dealt with easily. This is why most articles in the common agency literature conclude to the "failure" of the revelation principle.

The same thing happens here when information designers can communicate with the common receiver. The example below illustrates the limits of the revelation principle in the multi-senders common-receiver information design with elicitation. Hence the space of direct private persuasion

[16] This universal type space was constructed by Epstein and Peters [12].

mechanisms for sender D_i (which communication space is $\mathcal{M}_i = T$ for all $i = 1, \ldots, n$) is a priori not general enough to replicate all possible equilibria relative to more general feasible mechanisms.

Multiple senders competition and the limits of the revelation principle: an example

During a fair, a trial is proposed to a consumer (she). She can test two different but comparable products manufactured by two different firms. Then she has to publicly announce her favorite product, which is referred as voting for one product in what follows. Each product is offered to the consumer in limited supply: she is able to test a sample. Each product is indexed by $i = 1, 2$ and is of type $\omega_i \sim \mathcal{U}[0, 1]$. This type could represent the quality of the match between the product and the consumer or the inner quality of the good. The distribution $\mathcal{U}[0, 1]$ is the initial common prior of the game. I also assume that ω_1 and ω_2 are independent.

The consumer can choose how much to consume from each sample and which good to vote for. She does so to maximize her expected payoff. Her utility is given by

$$u(a_1, a_2, a_3, a_4, \omega_1, \omega_2) = \sum_{i=1}^{2} \left[(a_i + a_{i+2}) \left(\mathbf{1}_{\{\omega_i \geq \epsilon\}} - \kappa \mathbf{1}_{\{\omega_i < \epsilon\}}\right)\right] + f(\gamma)$$

and therefore her expected utility is given by

$$\mathbb{E}u(a_1, a_2, a_3, a_4, \omega_1, \omega_2) = (a_1 + a_3)(\mu_1 - \kappa(1 - \mu_1)) + (a_2 + a_4)(\mu_2 + \kappa(1 - \mu_2)) + f(\gamma)$$

where $\mu_i = \mathbb{P}(\omega_i \geq \epsilon)$. a_1 and a_2 are the quantities of good 1 and 2, respectively, she consumes. They are chosen in $[0, 1]$, one meaning that she consumes the whole sample, and zero that she does not consume at all. a_3 and a_4 are discrete choices: a_3, $a_4 \in \{0, 1\}$. They represent the voting choice of the consumer. If $a_{i+2} = 1$, the consumer votes for product i. The consumer can vote for at most one product, then $a_3 + a_4 \leq 1$. Furthermore the utility function is constructed so that the consumer do not enjoy poor quality products, so she will not consume a product of bad quality. Similarly she prefers to abstain rather than recommending a poor quality product. Finally, the receiver is assumed to have preferences on certain mechanisms being played.[17] This implies that, independently of the signal sent, the expected utility of the consumer can vary across offered mechanisms. For example, the consumer may enjoy the thrill of the unknown and get a utility premium when at least one firm

[17] This assumption is clearly non-canonical and is made to construct a counter example to the revelation principle in competing persuasion game with elicitation.

Table 1: Sample game and indirect mechanisms

	FR	NR
FR	$(\mu_0 - \frac{1}{4}\mu_0^2,\, \mu_0 - \frac{1}{4}\mu_0^2,\, 4\mu_0 - \mu_0^2)$	$(\mu_0,\, \frac{1}{2} + \frac{1}{2}(1-\mu_0),\, 7\mu_0 - 2\mu_0^2 - 2 + W)$
NR	$(\frac{1}{2} + \frac{1}{2}(1-\mu_0),\, \mu_0,\, 7\mu_0 - 2\mu_0^2 - 2 + W)$	$(\frac{3}{4},\, \frac{3}{4},\, 6\mu_0 - 3 + W)$

reveals no information at all.

Companies can only design tests to influence consumer's actions through her beliefs. In particular, there cannot be any transfer between the companies and the consumer.[18] Here we can think about these tests as what the presentation of the samples reveals about the products. I limit the companies' simple actions to $\{FR, NR\}$, where FR stands for full revelation and NR for no revelation. This constraint on the set of simple actions is for simplicity only. Then both firms simultaneously choose what experiment to pick among $\{FR, NR\}$ to maximize their expected payoff. The payoffs are

$$v_i(a_1, a_2, a_3, a_4, \omega_1, \omega_2) = \frac{1}{2}a_i + \frac{1}{2}a_{i+2}$$

Following our convention that ties in equilibria for the receiver are broken by Nash bargaining, and assuming that both firms have the same positive bargaining power, the optimal actions taken by the consumer are

$$\begin{aligned} a_i &= \mathbf{1}_{\{\mu_i \geq \frac{\kappa}{1+\kappa}\}}, \quad i = 1, 2, \\ a_j &= \mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_j = \mu_{-j} \geq \frac{\kappa}{1+\kappa}\}} \end{aligned}$$

The second action needs to be understood as the consumer voting for the best product and mixing with probability $\frac{1}{2}$ in case of tie.

For $\kappa = 1$ and $f(\gamma) = W$ if NR is played by at least one company, the above game is summarized in table 1 that shows all the payoffs.[19] When $W = 2 + \mu_0^2 - 3\mu_0$ and $\epsilon = \frac{7}{24}$, table 2 describes the payoffs of the game.

Here a direct mechanism is simply a probability distributions over simple actions as there is no private information. Then there is a single equilibrium in the above game in pure strategies and direct

[18] Although $f(\gamma)$ in the utility function of the consumer plays a role similar to a transfer.

[19] $\mu_0 = \mu_0^i$, $i = 1, 2$, is the prior probability of ω_i being greater than ϵ, i.e., $\mu_0 = 1 - \epsilon$.

Table 2: Payoffs of the sample game

	FR	NR
FR	$(\sim 0.5829, \sim 0.5829, \sim 2.2316)$	$(\frac{17}{24}, \frac{31}{48}^*, \sim 2.2316)$
NR	$(\frac{31}{48}^*, \frac{17}{24}, \sim 2.2316)$	$(\frac{3}{4}^*, \frac{3}{4}^*, \sim 1.6267)$

mechanism: (NR, NR). It yields the preferred outcome for the designers and the least preferred for the consumer. Furthermore there is no equilibrium in mixed strategies as NR is a dominant strategy for both sender.

However, if more complex mechanisms are allowed, new equilibrium outcomes can be supported. For example, if both information designers offer a menu $\{FR, NR\}$ and the consumer mixes between (FR, NR) and (NR, FR) with probability $\frac{1}{2}$,[20] it is a Bayesian Nash equilibrium. Hence it yields payoff $\frac{65}{96}$ to both firms. If either sender tries to deviate and forces action NR (no companies would ever force action FR as it is a dominated action), the consumer responds by choosing FR from the other firm's menu. Then the deviator gets a payoff of $\frac{31}{48} < \frac{65}{96}$.

This example suggests that new equilibria may appear when information designers can use more complex mechanisms. This results from the fact that the consumer (the common receiver) can play the role of a correlation device and can enforce new outcomes by punishing the deviator.

Here the new equilibria exist under a very particular condition: the consumer's payoff depends directly on the mechanism offered, which is a non-canonical assumption. This is similar to a transfer from the senders to the receiver to a certain extent.

However there is a priori no guarantee that something similar does not occur under more classic assumption, and in particular in the framework we define in section 3. Hence the existence of a new equilibrium when menus are allowed comes from the new roles the common receiver play. She becomes both a correlating device and a disciplinary device, that is she can induce equilibria that are profitable only when jointly played and she can also punish the deviating senders. So we need to find a way to overcome this difficulty. The solution comes from the common agency literature in mechanism design

[20]That she mixes with probability $\frac{1}{2}$ is an assumption. She could mix with any probability $p \in (0, 1)$.

5 Simplifying the problem

From the common agency literature and in particular Peters [25] and Martimort and Stole [20], we know that in common agency mechanism design, the difficulties resulting from the “failure" of the revelation principle for classic incentive compatible direct mechanisms can be partly overcome by looking at menus. This is sometimes referred as the “delegation principle" (see [20]). We show that, in information design with elicitation, the difficulties introduced by the competition in mechanisms among senders can be handled in a similar manner as the framework of information design with elicitation we developed is very close to mechanism design.

5.1 Menus and direct mechanisms

First we define menus and direct persuasion mechanisms. A menu is a private persuasion mechanism, which message space, for information designer D_i, $\mathcal{M}_i$ is a subset of the simple actions space: $\mathcal{M}_i \subset \mathcal{E}_i$. The contract associated with a menu mechanism is a mapping $\gamma_i : \mathcal{E}_i \to \mathcal{E}_i$.

Let Γ_i^M be the set of menus sender D_i can offer. A menu $\gamma_i \in \Gamma_i^M$ can be summarized by a measurable mapping

$$\gamma_i : \mathcal{E}_i \to \mathcal{E}_i$$

$$\pi_i \to \begin{cases} \pi_i \text{ if } \pi_i \in P \\ \bar{\pi}_i \text{ otherwise} \end{cases}$$

where P is a closed subset of $\mathcal{E}_i$ and $\bar{\pi}_i$ is an arbitrary element of P.

Direct persuasion mechanisms are defined as in [18]. They are the canonical private persuasion mechanisms, where the message space is the receiver's type space T. Let Γ_i^{DM} be the set of direct persuasion mechanism available to sender D_i. A direct persuasion mechanism $\gamma_i \in \Gamma_i^{DM}$ is characterized by a measurable mapping

$$\gamma_i : T \to \mathcal{E}_i$$

$$t \to \gamma_i(t)$$

A direct persuasion mechanism is said to be incentive compatible if it induces truth-telling as an optimal communication strategy, i.e., if and only if for all possible type $t \in T$, the common receiver R reports her type t truthfully. Since the receiver has no commitment power, it must imply that the expected payoff associated with a truthful report is weakly greater than the expected payoff associated with any other report. Let $\Gamma^{ICDM} = \prod_{i=n}^{n} \Gamma_i^{ICDM}$ be the set of incentive compatible direct persuasion mechanisms.

5.2 Two simplifying theorems

As noted previously, the universal type space is too complicated to be useful. If we want to find the equilibria of the competition in mechanisms among senders, we need to refine the set of mechanisms we look at. The two following theorems based on the common agency literature in mechanism design allow us to look only at incentive compatible direct mechanisms, as in the single designers case. They re-establish a revelation principle for multi-senders common-receiver persuasion games under the assumptions we made.[21]

The first theorem establishes, for any equilibrium relative to the set of universal persuasion mechanisms, the existence of an equilibrium in direct mechanisms that replicates all payoffs. It is based on theorem 1 in [20], theorem 2 in [25], and theorem 4 in [26].

Theorem 5.1. *Let $\{\gamma_1^*, \ldots, \gamma_n^*, c^*\} \in \Gamma^u \times \Delta(\mathcal{M}^u) \times \Delta(A)$ be an equilibrium relative to the universal persuasion mechanism space. Then there exists a (pure strategy) equilibrium in incentive compatible direct persuasion mechanism $\{\gamma_1', \ldots, \gamma_n', c'\} \in \Gamma^{DM} \times \Delta(T) \times \Delta(A)$ that preserves all payoffs.*[22]

Proof. In the Appendix 8.1.1. □

We provide the intuition for the proof of theorem 5.1 now. From the common agency literature, we know that any equilibrium relative to a set of mechanisms can be by an equilibrium relative to the set of menus. So we only have to show that there exists an equilibrium in incentive compatible direct mechanisms that preserves all payoffs for all equlibria relative to the set of menus.

To proof this claim, we construct an equilibrium in incentive compatible direct persuasion mechanisms that preserves all payoffs of an arbitrary equilibrium relative to the set of menus. Hence we

[21] They rely heavily on assumption 3, i.e., on the fact that persuasion mechanisms are payoff irrelevant. A sender is only affected by the strategy of another one through the action taken by the common receiver and the common receiver's actions do not depend on the mechanisms played directly but only on the beliefs they induce.

[22] Here $\{\gamma_1^*, \ldots, \gamma_n^*\}$ is assumed to include randomizations: the universal persuasion mechanisms space is large enough to include random mechanisms.

can think of a menu as a collection simple actions. When such a collection is proposed to the common receiver, she only picks the simple action that maximizes her expected utility given her type. So senders may as well propose only the a contract that associates to each type t the simple action a receiver R of type t would choose from menu offered in equilibrium, i.e, an incentive compatible direct mechanism. This preserves the distribution on receiver's actions played in equilibrium.

However when information designers offer direct mechanisms, new deviations may be profitable, as the common receiver cannot discipline the senders anymore. Fortunately she was already not before. This results from two features of our framework. First we assumed that the persuasion mechanisms were payoff irrelevant. Secondly the common receiver's expected utility is only a function of the mechanisms through her decision strategy since simple actions must be Bayes-plausible. So her choice from the menu of any designer D_i was "independent" of the mechanisms proposed by the other designers, D_{-i}. This allows us to replicate any equilibrium in menus.

The second theorem establishes that equilibria in incentive compatible direct persuasion mechanisms are robust when the principal can deviate to more complex mechanisms. It is based on lemma 3.1, theorem 6 in [25], and theorem 2 in [26].

We first define what it means to be weakly robust for an equilibrium in a mechanism game following Peters in [25].

Definition 3. *An equilibrium $(\gamma, c) \in \prod_{i=1}^{n}(\Gamma_i) \times \Delta(\mathcal{M}) \times \Delta(A)$ relative to a set of feasible mechanism Γ is said to be weakly robust if for any embedding α that maps Γ into Γ', $\alpha : \Gamma \to \Gamma'$ such that Γ' is compact metric, there exists an extension c' of c such that $(\alpha(\gamma), c')$ is an equilibrium on $\alpha(\Gamma) = \Gamma'$.*[23]

We now state our robustness theorem.

Theorem 5.2. *Let $\{\delta_1^*, \ldots, \delta_n^*, c\}$ be an equilibrium relative to the set of incentive compatible direct persuasion mechanism. Then there exists a weakly robust pure strategy equilibrium $\{\gamma_1^*, \ldots, \gamma_n^*, c^*\}$ relative to the set of incentive compatible direct persuasion mechanisms that is payoff equivalent to $\{\delta_1^*, \ldots, \delta_n^*, c\}$.*

Proof. In the Appendix 8.1.1. □

We try to present the intuition behind theorem 5.2. From the common agency literature again, we know that equilibria in menus are weakly robust. So theorem 5.2 tells us that we can extend this weak

[23] This is the definition given in [25]

robustness property to the equilibria in incentive compatible direct persuasion mechanisms. No sender can achieve a greater payoff in equilibrium by offering a menu rather than an incentive compatible direct mechanism. The intuition resemble the one behind theorem 5.1. Again the common receiver chooses only a simple actions from any menu. So in the end, any deviation in menu induce the distribution on actions as a deviation in direct mechanisms, thus the same expected payoff. However there is no profitable deviation in direct mechanism in the original equilibrium. Then there is also no profitable deviation in menus.

No communication The two theorems above are interesting in simplifying the problem we face, but also per se. Hence they imply that communication between the common receiver and the information designers is not necessarily needed. In particular, there is no reason for the senders to communicate with the receiver if the latter does not have any private information of his own. Even when the common receiver R has some private information, any communication about the "state of the market" and what the other designer proposes is useless. Senders can achieve any possible equilibrium as soon as they know the type of the receiver. This discussion is summarized in the next corollary.

Corollary 5.3. *Communication about the "state of the market" is not needed.*

5.3 Competition in direct mechanisms among senders

Recall that we are interested in the competition in persuasion mechanisms between the senders and in the display of information that results from this competition. We want to determine the strategic behaviors of the information designers. However, so far, the problem was too complex to tackle it.

We needed to determine the equilibria of a game in arbitrary mechanisms. In particular, we needed to find the set of Bayesian Nash equilibria of the normal form game between the senders in which the action space is Γ^u, defined by the continuation strategy (when it exists) of the common receiver. Fortunately the above subsection 5.2 allows us to simplify the study of competition among senders in information design with elicitation.

The aim of theorems 5.1 and 5.2 is to simplify the study of the competition in mechanisms game between the senders. They enable us to reduce, without loss of generality, the set of actions senders can take: from the set of universal persuasion mechanisms to the set of incentive compatible direct mechanisms. Furthermore theorem 5.2 guarantees that we are not creating new equilibria that would disappear when deviations to more complex mechanisms are allowed.

Therefore the problem of finding the equilibria of the multi-senders common-receiver game relative to the set of universal persuasion mechanisms reduces to the problem of finding the equilibria of the multi-senders common-receiver game relative to the set of incentive compatible direct persuasion mechanisms. This is a much simpler problem, which we are able to tackle in some cases. Section 6 presents an application in which we determine the equilibrium strategies of all players.

Before proceeding to this application, we present two existence criteria.[24] The first provides conditions for the existence of a Bayesian Nash equilibrium of the multi-senders common-receiver game when the common receiver has no private information of her own, i.e., $|T| = 1$. The second theorem extends this results to the case of a privately informed receiver.

First recall that an equilibrium of the game is a an array $(\gamma_1, \ldots, \gamma_n, c) \in \prod_{i=1}^n \Delta(\Gamma_i) \times \Delta(\mathcal{M}) \times \Delta(A)$ such that $c = (\tilde{m}, \tilde{a})$ is a continuation equilibrium given $(\gamma_1, \ldots, \gamma_n)$ and $(\gamma_1, \ldots, \gamma_n)$ is a Bayesian Nash equilibrium of the normal form game among sender defined by c. Hence the equilibrium continuation strategy of the common receiver (when it exists), c, induces a distribution on beliefs $\tau^\gamma(\mu, t)$ and therefore a distribution on actions $\tilde{a}^\gamma(t) = \tau^\gamma(\mu)\tilde{a}^c(\mu, t)$ conditional on the type $t \in T$ of the receiver and the array of mechanisms being played. These distributions on posterior beliefs and actions characterize the payoffs of the senders in the game in mechanisms they play. Information designer D_i's expected payoff is indeed

$$\bar{v}_i(\gamma) = \int_T \int_{\Delta(\Omega)} \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\tau^\gamma(\mu, t)\, dF(t)$$

Since the simple actions of any sender are Bayes-plausible distribution on posteriors on ω_i and the ω_i's are independent, the expected payoff of sender D_i rewrites

$$\bar{v}_i(\gamma) = \int_T \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^\gamma(t)(a)\, d\mu_0(\omega)\, dF(t)$$

Our equilibrium concept for the senders game is the Bayesian Nash equilibrium concept. So an equilibrium of the multi-senders common-receiver game relative to the set of incentive compatible direct persuasion mechanisms is defined by a Bayesian Nash equilibrium of the game in direct persuasion

[24]As pointed out by Frédéric Koessler, the payoff security of the game is not an assumption on the priors. So our two theorems do not establish existence per se. I am working on conditions that would guarantee the payoff security of the game.

mechanisms among senders and the associated continuation equilibrium for the common receiver.

We now state the first theorem. It is based on Reny's seminal paper on the existence of Nash equilibria in discontinuous games [28]. Note that an incentive compatible direct persuasion mechanism when the receiver has no private information of her own, i.e., $|T| = 1$, is a distribution on simple actions.

Theorem 5.4 provides conditions for the existence of an equilibrium of the multi-senders common-receiver game.

Theorem 5.4. *Under our assumptions, an equilibrium of the multi-senders common-receiver game relative to the set of direct persuasion mechanisms exists when the receiver has no private information of her own and the payoffs in the senders game are secure.*

Proof. In the Appendix 8.1.2. □

The proof relies on corollary 5.2 in [28]. We show that the normal form game defined by the continuation equilibrium of the common receiver is Hausdorff compact. Then we show that the game is reciprocally upper semicontinuous and conclude by Reny's corollary 5.2 (theorem 8.4).

From the proof of theorem 5.4, we can even be a little more precise. Hence, theorem 5.4 guarantees the existence of an equilibrium of the multi-senders common receiver game in which all senders play pure strategies.

Note that theorem 5.4 also guarantees the existence of an equilibrium in public persuasion mechanisms.

Corollary 5.5. *Under our assumptions, an equilibrium of the multi-senders common-receiver game relative to the set of public persuasion mechanisms exists.*

The second existence theorem accounts for the possibility that the receiver has private information of her own, i.e., that the type space has cardinality more than one. It provides conditions under which we believe that an equilibrium in the senders game, relative to the set of incentive compatible direct mechanisms, exist. Theorem 5.6 is based on Reny's corollary 5.2 and was also inspired by theorem 1 in [15].

Theorem 5.6. *Under our assumptions, an equilibrium of the multi-senders common-receiver game relative to the set of incentive compatible direct persuasion mechanisms exists when the payoffs in the senders game are secure.*

Proof. In the Appendix 8.1.2. □

The proof is similar to the proof of theorem 5.4 and also relies on corollary 5.2 in [28]. The main difference is in the way to prove that the sets of available actions to the senders in the normal form game defined by the continuation equilibrium of the common receiver are Hausdorff compact. Hence senders' actions are now proper mechanisms, and cannot be assimilated to the set of simple actions. This complicates the proof, since we now have to show compactness for the set of incentive compatible direct persuasion mechanisms. We deal with this issue by enlarging the feasible mechanisms set to the set of direct persuasion mechanisms Γ^{DM}. We show the compactness of this set under our assumptions using the reproducing kernel Hilbert space embedding for probability measures described in [29] and Fraňková-Helly selection theorem. Then we show that Γ^{ICDM} is closed in Γ^{DM}. The rest of the proof remains close to the above proof. We show that the game is reciprocally upper semicontinuous and conclude by Reny's corollary 5.2 (theorem 8.4).

Again this two theorems do not ensure the existence of an equilibrium a priori. They rather reassure us that an equilibrium of a particular game can be found when the payoff of the senders in the induced subgame between senders are secure. They are criteria of existence rather theorems of existence. Note also that the first existence theorem is actually a corollary of the second existence theorem.

From the above existence theorems and theorem 5.2, we extract one last corollary.

Corollary 5.7. *When the payoffs of the senders subgame induced by the continuation equilibrium played by the common receiver are secure, there exists a weakly robust equilibrium (relative to the set of universal mechanisms?).*

6 An application to a certification game

We propose a first application to our theory of multi-senders common-receiver in information design with elicitation. Multiple firms compete to be certified by an external regulatory agency. They are able to display information on their own products only through the design of "statistical experiments". However they cannot send any credible information on their rivals to the regulatory agency. We refer to this framework as a certification game.

6.1 Framework of the game

We study a certification game between two firms in the wood products industry that are trying to obtain an eco-certification from an external third party. Hence being certified has a positive impact on the price a firm can charge and on the demand it faces. This affirmation is motivated by two empirical studies on the wood products market in the US. As noted in [32], the environmental certification programs "are increasingly being recognized as significant market-based tools for linking manufacturing and consumer purchases". In particular, Vlosky et al. in [32] identifies a cluster of US consumers that would rather purchase certified wood products. Their findings are confirmed in [1]. The authors of the latter article find that some consumers are willing to pay up to a 10% premium for certified products. They also remark the potential existence of "niche markets [which] may potentially be exploited in the U.S." by certified firms in the wood industry. We construct our example to replicate the competition in information game that firms play in order to be certified, given a stylized market for wood products.

There are two wood companies, indexed by $j = 1, 2$. Both produce the same kind of wood. Their products are perfect substitutes. We are not interested in the market competition so we will assume that the prices of both firms' products are fixed exogenously to p. Marginal costs are zero. Firms can be environmentally-friendly or polluting. So the uncertain state of the world ω is a two-dimensional vector whose element are the status of the two firms: $\omega = (\omega_1, \omega_2) \in \{E, P\}^2$. E stands for environmentally-friendly and P for polluting. So there are four possible states of the world. Finally the regulator agency may have private information.

The certification game occurs prior to commercialization. In this game, the regulatory agency (she) can give an eco-certification to each firm according to a known decision rule (that maximizes her expected utility). The firms (that are the information designers) can display information on the status of their supply chain to the regulator by designing a "statistical experiment" that credibly reveal some information on the state of the world. However they do not know in advance whether they would be perceived as environmentally-friendly or polluting. This assumption may rely on the absence of full control of the board on the supply chain for example. The regulatory agency is only willing to certify environmentally-friendly firms. Finally all the players share a common prior on the status of the firms $\mu_0 = (\mu_0^1, \mu_0^2) = (\mu_0, \mu_0)$. ω_1 and ω_2 are independent random variable and $\mu_0 = \mathbb{P}(\omega_j = E)$.

The timing of the game is as follows.

- At time $t = 0$, nature draws ω. No player observe the draw.
- At time $t = 1$, two envoys from the regulator agency are sent to the firms. Both firms, $j = 1, 2$, simultaneously design "statistical experiment" to credibly display information on their status to the envoy.
- At time $t = 2$, signals are generated from the "statistical experiments". They are reported truthfully by the envoys to the regulator agency. The latter updates her beliefs on ω using Bayes law.
- At time $t = 3$, the regulator agency chooses whether to certify firm $j = 1, 2$ given her new beliefs on ω. She takes action $a_j \in A = \{0, 1\}$, $j = 1, 2$, where $a_j = 1$ means that the regulator agency provide a eco-certification to firm j.
- At time $t = 4$, both products are commercialized. Consumers only observe whether a product is certified by the regulator agency.

We want to focus on the competition in persuasion mechanisms. So we mostly abstract from the consumers behavior. We assume that the total demand for wood products at price p is $D(p)$. It is normalized to 1. When a product hits the market, consumers buy it if and only if it is certified. Then the demand for wood products of firm j is zero if firm j is not certified, $\frac{1}{2}$ if both firms are certified, and 1 if firm j is certified but firm $-j$ is not.

6.2 The non competitive case

We first study a case in which the agency's decision rule defines a non-competitive non-strategic game in mechanisms between the two wood companies. It serves as a reference point.

Let the regulator's payoff be

$$u(a_1, a_2, \omega_1, \omega_2) = \sum_{j=1}^{2} a_j \left(\mathbf{1}_{\{\omega_j = E\}} - \kappa \mathbf{1}_{\{\omega_j = P\}} \right)$$

where $a_j \in \{0, 1\}$ and $a_j = 1$ corresponds to the decision to certify the products of firm j. κ is the regulator's type. It represents her preference on certifying a polluting firm rather than not to certify a environmentally-friendly firm. Here, we assume that κ is commonly known. The regulatory agency has no private information. ω_j is the status of firm j. It is unobserved, so the regulator agency maximizes

$$\begin{aligned}\hat{u}(a_1, a_2, \mu_1, \mu_2) &= \mathbb{E}_{(\mu_1,\mu_2)} u(a_1, a_2, \omega_1, \omega_2) \\ &= \sum_{j=1}^{2} a_j \left(\mu_j - \kappa(1-\mu_j)\right) \\ &= \sum_{j=1}^{2} a_j \left((1+\kappa)\mu_j - \kappa\right)\end{aligned}$$

where $\mu_j = \mathbb{P}(\omega_j = E)$, $j = 1, 2$. Therefore the regulator agency preferred actions are

$$a_j^*(\mu_j) = \mathbf{1}_{\{\mu_j \geq \frac{\kappa}{1+\kappa}\}}$$

In particular, a_j^* is independent of μ_{-j}. Recall that, under our assumptions, ω_1 and ω_2 are independent and firm j's signal is only informative about ω_j. This implies that firm j can only influence μ_j. So the game is not strategic, as firm j cannot influence a_{-j}^*.

The wood companies' payoffs are given by

$$v_j(a, \omega) = pD_j(a, \omega), \quad j = 1, 2$$

where $D_j(a, \omega)$ is the demand firm j faces given her status and the agency's actions. From the above discussion,

$$D_j(a, \omega) = \begin{cases} 1 \text{ if } a_j = 1 \neq a_{-j} \\ \frac{1}{2} \text{ if } a_j = 1 = a_{-j} \\ 0 \text{ if } a_j = 0 \end{cases} \qquad j = 1, 2$$

Finally companies' expected payoff given beliefs μ_j and μ_{-j} are

$$\hat{a}_j(\mu) = \begin{cases} p \text{ if } \mu_j \geq \frac{\kappa}{1+\kappa} > \mu_{-j} \\ \frac{p}{2} \text{ if } \mu_j, \mu_{-j} \geq \frac{\kappa}{1+\kappa} \\ 0 \text{ if } \mu_j < \frac{\kappa}{1+\kappa} \end{cases} \qquad j = 1, 2$$

Firms compete in mechanisms. From theorem 5.1 and 5.2, we know that we can focus on incentive compatible direct mechanisms, and since the receiver has no private information in this example, competition in mechanisms boils down to competition in simple actions. Furthermore, from theorem 5.4, an equilibrium exists.[25] So we compute it.

The strategic interactions in this example are very limited, as the optimal actions of both senders are independent of the action that the other can take, and the objective of both firms is simply to induce beliefs $\mu_j \geq \frac{\kappa}{1+\kappa}$ as often as possible. This is then very similar to the judge prosecutor example in [16].

Proposition 6.1. *When $\mu_0 \geq \frac{\kappa}{1+\kappa}$, it is an equilibrium for both firms to play no revelation. When $\mu_0 < \frac{\kappa}{1+\kappa}$, the unique Bayesian Nash equilibrium of the game in simple actions between the two wood producers is $(\tau_1^* = (\frac{\mu_0(1+\kappa)}{\kappa}, \frac{\kappa}{1+\kappa}, 0), \tau_2^* = (\frac{\mu_0(1+\kappa)}{\kappa}, \frac{\kappa}{1+\kappa}, 0))$, where $\tau_j^* \in \Delta(\Omega_j)$ is the strategy played by firm j in equilibrium. It is described by the triplet $(\lambda_j, \mu_j', \mu_j'')$ where λ_j is the probability that the chosen "experiments" induces μ_j' and $1 - \lambda_j$ is the probability that the chosen "experiments" induces μ_j''.*

Proof. In the Appendix 8.2. □

The "statistical experiments" chosen by the two firms produce only two different signals and so are fully described by the above triplet. This is as in Kamenica and Gentzkow [16].

In the non competitive case, although there are multiple senders, they do not interact strategically. So all the results of information design with a single sender remains true. In particular, it is still sufficient for the firms to produce two signals only.

6.3 The competitive case

The analysis of the competitive case is very close to the competition in information disclosure studied by Au and Kawai in a recent paper [4].[26] The math could even be an application of their more general framework since the techniques we employ are similar; although their proofs rely more on the concavification result of Kamenica and Gentzkow in [], while my proofs are more constructive and close to the application I consider. However, we employ the same techniques borrowed from the study of all pay auctions with complete information pioneered in [5]. See also the computation of the equilibrium of the wage posting game in [9].

[25] The payoffs of the senders game are obviously secure here.

[26] I thank Frédéric Koessler for pointing it to my attention.

We still include the competitive example in which the receiver has no private information. We do so for two reasons. The first is that we develop this analysis independently. The second is that it should be understood as the basis of a coming extension. Hence I am currently working on a related multi-sender common-receiver model of information design with elicitation in which κ is privately known to the receiver.

This subsection modifies the utility function of the regulator agency, and therefore her decision rule, to induce a strategic game in mechanisms between the two wood producers. It constructs a utility function for the receiver so that she is not willing to certify the products of both firms. The reason for the regulator agency to certify only one firm could be reputation for example. Delivering the eco-certification to too many firm may hurt the credibility of the certifier. The regulator's utility is

$$u(a_1, a_2, \omega_1, \omega_2) = \sum_{j=1}^{2} a_j \left(\mathbf{1}_{\{\omega_j = E\}} - \kappa \mathbf{1}_{\{\omega_j = P\}} - a_{-j}\right)$$

where her available actions are still $a_j \in \{0, 1\}$. $a_j = 1$ corresponds to the decision to certify the products of firm j. κ is the regulator's type. It represents her preference on certifying a polluting firm rather than not to certify a environmentally-friendly firm. Here, we assume that κ is commonly known. ω_j is the status of firm j. It is unobserved, so the regulator agency maximizes

$$\begin{aligned}\hat{u}(a_1, a_2, \mu_1, \mu_2) &= \mathbb{E}_{(\mu_1, \mu_2)} u(a_1, a_2, \omega_1, \omega_2) \\ &= \sum_{j=1}^{2} a_j \left((1+\kappa)\mu_j - \kappa - a_{-j}\right)\end{aligned}$$

where $\mu_j = \mathbb{P}(\omega_j = E)$, $j = 1, 2$. Therefore the decision strategy[27] of the regulator agency is

$$a_j^*(\mu_j) = \begin{cases} 1 \text{ if } \mu_j > \mu_{-j} \text{ and } \mu_j \geq \frac{\kappa}{1+\kappa} \\ \frac{1}{2} \text{ if } \mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa} \\ 0 \text{ otherwise} \end{cases}$$

where the second case should be understood as mixing with probability $\frac{1}{2}$ between actions $a_1 = 1$ and $a_2 = 2$. The regulator has no private information, so there is no communication, since, by theorem

[27]The strategy takes into account the bargaining subgame.

5.1 and 5.2, we can focus on competition in direct persuasion mechanisms. So direct persuasion mechanisms boil down to simple actions.

Recall also that the prior beliefs of the regulator agency is $\mu_0 = \mathbb{P}(\omega_1 = E) = \mathbb{P}(\omega_2 = P)$.[28]

In order to find the equilibria of the certification game, we need to study the game between the two companies induced by the continuation equilibrium played by the regulator. In this game, the set of actions available to firm j is the set of Bayes-plausible distribution on ω_j it can induce: $\mathcal{E}_i \subset \Delta(\Delta(\Omega_i))$. Recall that the payoffs functions of the firms $j = 1, 2$, are

$$v_j(a, \omega) = pD_j(a, \omega), \quad j = 1, 2$$

where $D_j(a, \omega)$ is the demand firm j faces given her status and the agency's actions. From the above discussion,

$$D_j(a, \omega) = \begin{cases} 1 \text{ if } a_j = 1 \neq a_{-j} \\ \frac{1}{2} \text{ if } a_j = 1 = a_{-j} \\ 0 \text{ if } a_j = 0 \end{cases}$$

Therefore the expected payoff of the two companies that characterize the game induced by the regulator agency's decision rule are

$$\bar{v}_j(\tau_j, \tau_{-j}) = \mathbb{E}_{(\tau_1(\mu_1), \tau_2(\mu_2))} \hat{v}_j(\mu_1, \mu_2)$$

where

$$\hat{v}_j(\mu_1, \mu_2) = p\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - p\frac{1}{2}\mathbf{1}_{\{\mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa}\}}$$

Then

$$\bar{v}_j(\tau_j, \tau_{-j}) = p\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)\right)\left[\sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu) + \left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right)\right.$$
$$\left. \times \int_{\Delta(\Omega_j)} \int_{\Delta(\Omega_{-j})} \left(\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa}\}}\right) d\tau_{-j}(\mu_{-j}) d\tau_j(\mu_j)\right]$$

[28] Authorizing the prior beliefs for ω_1 and ω_2 to be different does not change the analysis below.

where $\tau_j \in \mathcal{E}_j$ is the cumulative distribution function of beliefs induced by firm j on ω_j. We identify the strategy played by firm j and the distribution on beliefs it induces. Recall that it must be a Bayes-plausible distribution on posteriors beliefs.

We look for the equilibria of the firms subgame. Note that the two companies' payoffs are secure, since they are continuous in τ_j by Lebesgue dominated convergence theorem. Then theorem 5.4 guarantees the existence of an equilibrium of the companies subgame. Furthermore from lemma 3.1, we can look for pure strategies equilibria without loss of generality. However to compute the equilibria of the above game remains a tricky exercise. We use a method close to the one proposed in [9] and [8] to compute the equilibrium solution of the wage posting game. Recall that μ_j is the probability that the wood producer is environmentally-friendly. So it is single-dimensioned and belongs to $[0, 1]$. Thus we look for Bayes-plausible distributions, τ_1, τ_2, on $[0, 1]$ that constitute an equilibrium of the game between firms.

We show that no distribution in equilibrium puts a positive mass in $(0, \frac{\kappa}{1+\kappa})$. So the total probability mass below $\frac{\kappa}{1+\kappa}$ is at zero in equilibrium.

Lemma 6.2. *In equilibrium,* $\tau_j(0) = \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)$, $j = 1, 2$.

Proof. Suppose that $\tau_j(0) \neq \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)$ in equilibrium. Then $\tau_j(0) < \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)$ and there exists $[a, b] \subset (0, \frac{\kappa}{1+\kappa})$ such that $\tau_j(b) - \tau_j(a) > 0$. When firm j plays this strategy, it gets

$$\bar{v}_j(\tau_j, \tau_{-j}) = p \left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)\right) \left[\sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu) + \left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right)\right.$$
$$\left. \times \int_{\Delta(\Omega_j)} \int_{\Delta(\Omega_{-j})} \left(\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa}\}}\right) d\tau_{-j}(\mu_{-j}) d\tau_j(\mu_j)\right]$$

Consider now the strategy τ_j' that moves some weight $\epsilon > 0$ away from $[a, b]$ to $\frac{\kappa}{1+\kappa}$ and some weight $\epsilon' > 0$ away from $[a, b]$ to zero, so that the new strategy is still Bayes-plausible. Such ϵ and ϵ' always exist since $\tau_j(b) - \tau_j(a) > 0$. We claim that this is a profitable deviation. Hence the payoff of firm j

is now

$$\bar{v}_j(\tau_j', \tau_{-j}) \geq p\left(1+\epsilon - \sup_{\mu<\frac{\kappa}{1+\kappa}} \tau_j(\mu)\right)\left[\sup_{\mu<\frac{\kappa}{1+\kappa}} \tau_{-j}(\mu) + \left(1 - \sup_{\mu<\frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right)\right.$$
$$\left. \times \int_{\Delta(\Omega_j)}\int_{\Delta(\Omega_{-j})}\left(\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_1=\mu_2 \geq \frac{\kappa}{1+\kappa}\}}\right) d\tau_{-j}(\mu_{-j}) d\tau_j(\mu_j)\right]$$
$$> \bar{v}_j(\tau_j, \tau_{-j})$$

So (τ_j, τ_{-j}) cannot be an equilibrium. □

We show that no equilibrium distribution has a mass point in $[\frac{\kappa}{1+\kappa}, 1)$. So the distributions τ_j, $j = 1, 2$, are continuous on $[0, 1)$.

Lemma 6.3. *τ_j is continuous on $[0, 1)$, $j = 1, 2$.*

Proof. From lemma 6.2, we know that there is no mass point in $(0, \frac{\kappa}{1+\kappa})$ so τ_j is continuous on $[0, \frac{\kappa}{1+\kappa})$, $j = 1, 2$. We show that there is no mass point on $[\frac{\kappa}{1+\kappa}, 1)$ too. Let (τ_j, τ_{-j}) be an equilibrium such that τ_j is not continuous on $(0, 1)$ in equilibrium. There exists a mass point $\bar{\mu} \in [\frac{\kappa}{1+\kappa}, 1)$ such that $\tau_j(\bar{\mu}) > \sup_{\mu<\bar{\mu}} \tau_j(\mu)$ an we can construct a profitable deviation.

Distinguish three cases.

- Suppose that there exists $r > 0$ such that $(\bar{\mu} - r, \bar{\mu}] \subset [\frac{\kappa}{1+\kappa}, 1]$ and $\tau_{-j}(\bar{\mu}) - \tau_{-j}(\bar{\mu} - r) > 0$, where r is small. Then we show there exists ϵ, ϵ' such that moving some faction $\epsilon > 0$ of the mass to $\bar{\mu}' = \inf_{\mu>\bar{\mu}} \mu$ and some fraction ϵ' of the mass to $\bar{\mu} - r$ (to preserve Bayes plausibility) is a profitable deviation. Call this new strategy τ'_{-j}. Then

$$\begin{aligned}
\bar{v}_j(\tau'_j, \tau_{-j}) - \bar{v}_j(\tau_j, \tau_{-j}) &= p\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)\right)\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right) \\
&\times \Bigg[\int_{\Delta(\Omega_j)}\left(\int_{\Delta(\Omega_{-j})}\left(\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa}\}}\right) d\tau'_{-j}(\mu_{-j})\right. \\
&- \left.\int_{\Delta(\Omega_{-j})}\left(\mathbf{1}_{\{\mu_j \geq \mu_{-j} \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu_1 = \mu_2 \geq \frac{\kappa}{1+\kappa}\}}\right) d\tau_{-j}(\mu_{-j})\right) d\tau_j(\mu_j)\Bigg] \\
&= p\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)\right)\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right) \\
&\times \int_{\Delta(\Omega_j)}\Bigg(\tau'_{-j}(\mu_j) - \frac{1}{2}(\tau'_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau'_j(\mu)) \\
&- \tau_{-j}(\mu_j) + \frac{1}{2}(\tau_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau_{-j}(\mu))\Bigg)\mathbf{1}_{\{\mu_j \geq \frac{\kappa}{1+\kappa}\}} d\tau_j(\mu_j) \\
&= p\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_j(\mu)\right)\left(1 - \sup_{\mu < \frac{\kappa}{1+\kappa}} \tau_{-j}(\mu)\right) \\
&\times \int_{\bar{\mu}-r}^{\bar{\mu}+r}\left(\tau'_{-j}(\mu_j) - \frac{1}{2}(\tau'_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau'_j(\mu)) - \tau_{-j}(\mu_j) + \frac{1}{2}(\tau_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau_{-j}(\mu))\right) d\tau_j(\mu_j)
\end{aligned}$$

Then

$$\begin{aligned}
&\bar{v}_j(\tau'_j, \tau_{-j}) - \bar{v}_j(\tau_j, \tau_{-j}) > 0 \\
\Leftrightarrow \int_{\bar{\mu}-r}^{\bar{\mu}+r}&\left(\tau'_{-j}(\mu_j) - \frac{1}{2}(\tau'_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau'_j(\mu)) - \tau_{-j}(\mu_j) + \frac{1}{2}(\tau_{-j}(\mu_j) - \sup_{\mu < \mu_j} \tau_{-j}(\mu))\right) d\tau_j(\mu_j) > 0
\end{aligned}$$

Note that the above inequality is implied by

$$-\epsilon'\left(\sup_{\mu < \bar{\mu}} \tau_j(\mu) - \tau_j(\bar{\mu} - r)\right) + \epsilon\left(\tau_j(\bar{\mu}) - \sup_{\mu < \bar{\mu}} \tau_j(\mu)\right) > 0$$

From Bayes-plausibility, we have

$$m \equiv \int_{\bar{\mu}-r}^{\bar{\mu}} \mu \, d\tau_{-j}(\mu) = \epsilon'(\bar{\mu} - r) + \epsilon\bar{\mu}$$

Then

$$-\epsilon'\left(\sup_{\mu<\bar{\mu}}\tau_j(\mu)-\tau_j(\bar{\mu}-r)\right)+\epsilon\left(\tau_j(\bar{\mu})-\sup_{\mu<\bar{\mu}}\tau_j(\mu)\right)>0$$
$$\Leftrightarrow \epsilon > \frac{pm}{\lambda(\bar{\mu}-r)+p\bar{\mu}}$$

where $\lambda = \tau_j(\bar{\mu}) - \sup_{\mu<\bar{\mu}} \tau_j(\mu)$ and $p = \sup_{\mu<\bar{\mu}} \tau_j(\mu) - \tau_j(\bar{\mu}-r)$. For r small enough, we can always find ϵ and ϵ' in $(0,1)$ such that the inequality holds. Then firm $-j$ has a profitable deviation, and we reach a contradiction. So there cannot be an equilibrium in which there is a mass point in $(\frac{\kappa}{1+\kappa},1)$ and the equilibrium strategy of the other firm puts a positive mass on an interval preceding this mass point.

- Suppose that there exists $r>0$ such that $(\bar{\mu}-r,\bar{\mu}]\subset[\frac{\kappa}{1+\kappa},1]$ and $\tau_{-j}(\bar{\mu})-\tau_{-j}(\bar{\mu}-r)=0$. Then there is an obvious profitable deviation for player j: move the mass point $\bar{\mu}$ to some $\mu\in(\bar{\mu}-r,\bar{\mu})$ and increase its mass to preserve Bayes-plausibility. This is a profitable deviation. So there cannot be an equilibrium in which there is a mass point in $(\frac{\kappa}{1+\kappa},1)$ and the equilibrium strategy of the other firm puts no mass on an interval preceding this mass point.
- Suppose that τ_j has a mass point at $\frac{\kappa}{1+\kappa}$ and τ_{-j} is strictly increasing on $[\frac{\kappa}{1+\kappa},\frac{\kappa}{1+\kappa}+r)$ for some r. Then there exists $\epsilon>0$ such that it is a profitable deviation for player j to split some of the weight of the mass point between 0 and $\frac{\kappa}{1+\kappa}+\epsilon$. So there cannot be a mass point at $\frac{\kappa}{1+\kappa}$ in equilibrium.

Therefore there is no mass point in $[\frac{\kappa}{1+\kappa},1)$ and τ_j,τ_{-j} are continuous on $(0,1)$ in equilibrium. □

We show that supp $d\tau_j=$ supp $d\tau_{-j}$, where $d\tau_j$ in a non-rigorous notation indicates the points in $[0,1]$ on which supp τ_j puts a positive mass and the points in the support of the probability density function associated with the strategy of j when the cumulative distribution function is continuous.

Lemma 6.4. *In equilibrium supp $d\tau_1=$ supp $d\tau_2$.*

Proof. From lemma 6.2, we already know that both strategies puts all mass below $\frac{\kappa}{1+\kappa}$ at zero. So we only have to show that it holds on $[\frac{\kappa}{1+\kappa},1]$. We prove it by contradiction: suppose that supp $d\tau_j\neq$ supp $d\tau_{-j}$ on $[\frac{\kappa}{1+\kappa},1]$. Distinguish three cases.

- Suppose that $\hat{\mu}_j=\sup\text{supp}\ d\tau_j\neq\hat{\mu}_{-j}=\sup\text{supp}\ d\tau_{-j}$. Without loss of generality, assume that $\hat{\mu}_j>\hat{\mu}_{-j}$. Then firm j has a profitable deviation. Hence it is profitable to move all the mass there currently is on the interval $(\hat{\mu}_{-j},\hat{\mu}_j]$ to $\mu'=\inf_{\mu>\hat{\mu}_{-j}}\mu$ and increase the probability to play μ' to restore Bayes-plausibility. Therefore, in equilibrium, we must have $\hat{\mu}_{-j}=\hat{\mu}_j$.
- Suppose that $\check{\mu}_j=\inf\text{supp}\ d\tau_j\neq\check{\mu}_{-j}=\inf\text{supp}\ d\tau_{-j}$. Without loss of generality, assume that

$\check{\mu}_j > \check{\mu}_{-j}$. Then firm $-j$ puts no mass on $[\frac{\kappa}{1+\kappa}, \check{\mu}_j)$ and $\check{\mu}_{-j} = \frac{\kappa}{1+\kappa}$ for otherwise there is a profitable deviation. Hence moving all mass in $[check\mu_{-j}, \check{\mu}_j)$ to $\frac{\kappa}{1+\kappa}$ and increasing the probability to play $\frac{\kappa}{1+\kappa}$ to restore Bayes-plausibility would be a profitable deviation. This implies that there is a mass point at $\frac{\kappa}{1+\kappa}$. Now consider the strategy of firm j and note that moving some mass ϵ from $[\check{\mu}_j, \check{\mu}_j + r]$, for r small, to $\mu' = \inf_{\mu > \frac{\kappa}{1+\kappa}} \mu$ is a profitable deviation. Hence $\tau_{-j}(\check{\mu}_j + r) = \tau_{-j}(\frac{\kappa}{1+\kappa}) + \tau_{-j}(\check{\mu}_j + r) - \tau_{-j}(\check{\mu}_j)$. Choosing ϵ and r such that $\tau_{-j}(\check{\mu}_j + r) - \tau_{-j}(\check{\mu}_j) = \epsilon(\check{\mu}_j - \frac{\kappa}{1+\kappa})$ yields a profitable deviation. So there cannot be an equilibrium in which $\check{\mu}_j = \inf \operatorname{supp} d\tau_j \neq \check{\mu}_{-j} = \inf \operatorname{supp} d\tau_{-j}$.

- Suppose now that there exists an interval $[a, b] \subset [\check{\mu}_j, \hat{\mu}_j] = [\check{\mu}_{-j}, \hat{\mu}_{-j}] \subset [\frac{\kappa}{1+\kappa}, 1]$ such that $\tau_j(a) - \tau_j(b) > 0$ and $\tau_{-j}(a) - \tau_{-j}(b) = 0$ in equilibrium. Then moving all the mass on $[a, b]$ to $\mu' = \inf_{\mu > a} \mu$ and increasing the probability to play μ' to restore Bayes-plausibility is a profitable deviation for firm j. So there cannot be an equilibrium in which one firm may play some posterior μ and not the other.

Thus, in equilibrium, the union of the set of mass points, played under strategy τ_j and τ_{-j}, and the support of the probability density function associated with the strategies τ_j and τ_{-j}, when differentiable, are identical for both firms, i.e., supp $d\tau_j =$ supp $d\tau_{-j}$ in equilibrium. □

We show that supp $d\tau_j \cap [\frac{\kappa}{1+\kappa}, 1)$ is connected. This implies that there exists an interval $[\frac{\kappa}{1+\kappa}, \mu_{max}]$ such that τ_j is strictly increasing on $[\frac{\kappa}{1+\kappa}, \mu_{max}]$.

Lemma 6.5. *In equilibrium, supp* $d\tau_j \cap [\frac{\kappa}{1+\kappa}, 1)$, $j = 1, 2$, *is connected.*

Proof. The proof constructs a profitable deviation following exactly the reasoning of the second • in the above proof.

Suppose that there exists an interval $[a, b] \subset [\frac{\kappa}{1+\kappa}, \hat{\mu}]$ such that $\tau_j(b) - \tau_j(b) = \tau_{-j}(b) - \tau_{-j}(b) = 0$. Then it is a profitable deviation for firm j, for example, to move some of the mass it puts on $[b, b + r]$ to $\mu' = \underset{\mu > a}{\mu}$. So τ_j and τ_{-j} are strictly increasing on $[\frac{\kappa}{1+\kappa}, \hat{\mu}]$. □

From lemmas 6.2 to 6.5, we obtain a quite precise characterization of the strategies in equilibrium. Hence any strategy played by firm j, $j = 1, 2$, in equilibrium must satisfy

$$\tau_j(\mu) = \begin{cases} p_{0,j} \text{ if } \mu \in [0, \frac{\kappa}{1+\kappa}) \\ p_{0,j} + G_j(\mu) \text{ if } \mu \in [\frac{\kappa}{1+\kappa}, \hat{\mu}] \\ p_{0,j} + G_j(\hat{\mu}) \text{ if } \mu \in [\hat{\mu}, 1) \\ 1 \text{ if } \mu = 1 \end{cases}$$

where $G(\mu)$ is a strictly increasing continuous function such that $G_j(\frac{\kappa}{1+\kappa}) = 0$ and $G_j(\hat{\mu}) \leq 1 - p_0$. Furthermore τ_j must also be Bayes-plausible. Finally, for τ_1, τ_2 to be an equilibrium, τ_1 must be a best-response to τ_2 and vice-versa. Hence any strategy τ_j must keep the other player indifferent between all $\mu_{-j} \in \text{supp } d\tau_{-j} \cap [\frac{\kappa}{1+\kappa}, 1)$. From this observation, we can compute the equilibrium strategies for both firms: τ_1 and τ_2.

Hence if the "indifference" condition on $[\frac{\kappa}{1+\kappa}, 1]$ was not respected, there would be no reason for the firms to induce all these posterior beliefs. In particular, they could deviate to a signal technology with two signals that puts a maximal mass to the preferred posterior. The companies only induce posteriors that maximize their expected payoff in equilibrium and therefore must be indifferent between all the induced posterior above $\frac{\kappa}{1+\kappa}$. Bayes-plausibility is always guaranteed by the signal induced beliefs $\mu_j = 0$.

Then $\forall \mu \in \text{supp } \tau_{-j} \cap [\frac{\kappa}{1+\kappa}, 1)$

$$\begin{aligned}
&p\frac{\mu_0}{\mu}\left(p_{0,j} + (1-p_{0,j})\int_{\frac{\kappa}{1+\kappa}}^{\hat{\mu}} (\mathbf{1}_{\{\mu \geq \mu_j \wedge \frac{\kappa}{1+\kappa}\}} - \frac{1}{2}\mathbf{1}_{\{\mu = \mu_j \geq \frac{\kappa}{1+\kappa}\}})\, d\tau_j(\mu_j)\right) \\
&= p\frac{\mu_0}{\mu}\left[p_{0,j} + (1-p_{0,j})\frac{G_j(\mu)}{G_j(\hat{\mu})}\right] \\
&= p\frac{\mu_0(1+\kappa)}{\kappa}p_{0,j} \\
&= p\frac{\mu_0}{\hat{\mu}}\left[p_{0,j} + (1-p_{0,j})\right]
\end{aligned}$$

from the indifference remark above and the Bayes-plausibility constraint. The first equality comes from conditioning on $\mu > \frac{\kappa}{1+\kappa}$, the second equality comes from evaluating the above expression at $\frac{\kappa}{1+\kappa}$, and the last equality comes from evaluating the above expression at $\hat{\mu}$. Then, $\forall \mu \in \text{supp } \tau_{-j} \cap [\frac{\kappa}{1+\kappa}, 1)$,

$$\begin{aligned}
G_j(\mu) &= G_j(\hat{\mu})\left[\frac{\mu\frac{1+\kappa}{\kappa}p_{0,j} - p_{0,j}}{1 - p_{0,j}}\right] \\
&= G_j(\hat{\mu})\frac{\mu - \frac{\kappa}{1+\kappa}}{\frac{\kappa}{p_{0,j}(1+\kappa)} - \frac{\kappa}{1+\kappa}}
\end{aligned}$$

and

$$\hat{\mu} = \frac{\kappa}{p_{0,j}(1+\kappa)}$$

In equilibrium, posterior beliefs are therefore uniformly distributed on $\left[\frac{\kappa}{1+\kappa}, \frac{\kappa}{(p_{0,j}+p_{\frac{\kappa}{1+\kappa},j})(1+\kappa)}\right]$.

So two strategies τ_j and τ_{-j} constitute an equilibrium if and only if they satisfy the following conditions. They must be Bayes-plausible:

$$\mu_0 = G_j(\hat{\mu})\frac{\frac{\kappa}{1+\kappa}+\hat{\mu}}{2} + (1 - G_j(\hat{\mu}) - p_{0,j}), \quad j = 1, 2,$$

They must satisfy an "indifference" condition at $\hat{\mu}$:

$$\hat{\mu} = \frac{\kappa}{(1+\kappa)p_{0,j}}, \quad j = 1, 2,$$

Finally they must satisfy a last "indifference" condition at 1:

$$p\mu_0\left(1 - \frac{p_{1,j}}{2}\right) = p\mu_0\frac{1+\kappa}{\kappa}p_{0,j} \;\text{ or }\; p_{1,j} = 0, \quad j = 1, 2,$$

where $p_{1,j} = 1 - p_{0,j} - p_{\frac{\kappa}{1+\kappa},j} - G_j(\mu)$. So to determine the equilibria of the game between firms, we need to find the parameters that solves these equations. This is done in the proof of the proposition below.

Proposition 6.6. *The information design game admits a unique equilibrium. This equilibrium is symmetric. Furthermore, in equilibrium, the wood producers always display some information. If* $\mu_0 < \frac{2}{1+\kappa}$*, partial revelation is optimal. If* $\mu_0 \geq \frac{2}{1+\kappa}$*, full revelation is optimal.*

Proof. In the Appendix 8.2. □

In the competitive equilibrium between firms, more information is disclosed than in a non-competitive equilibrium. Hence the the competitive equilibrium strategies of the senders are weakly more informative than their non-competitive equilibrium strategies.

Furthermore it is worth noting another last point. Going back from beliefs to signals, in the competitive framework, although there is only two different state and the world and two different actions, the equilibrium strategies are supported by a continuum of signals. This departs from the result in the single designer case. Hence it is no longer sufficient to display no more signals in equilibrium than the maximum between the cardinality of the state space and the cardinality of the action set. However, as in the single information designer case, no good type sends a bad signal. The difference occurs when sending good signals. Information designers tend to increase how revealing this signals are to increase their chances to be selected.

From an individual sender's perspective, each induced posterior is associated with a probability of being certified. Increasing the posterior value of being environmentally-friendly gives a higher probability of being certified. However, the higher this probability is, less likely to arise it because of the Bayes-plausibility constraint. Thus each sender's optimal disclosure has to balance this trade-off.

7 Extensions

As mention above, some work still needs to be done. First I am currently trying to determine conditions on the priors of the game that would ensure that the payoffs of the induced subgame between the information designers are secure. Secondly I am working on an extension of the application in which the preference parameter κ is privately known to the receiver. We believe that the equivalence between private persuasion and public persuasion established by Kolotilin et al. in [18] also holds in this case. Finally, it would be interesting to relax the independence assumption between the dimension of the state of the world. From [14], we know that the characterization of the set of equilibria is simple when the dimension are perfectly correlated, but we do not know what would happen under imperfect correlation.

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8 Appendix

8.1 Appendix for section 5

8.1.1 Proofs of the two simplifying theorems

We start by reproducing the results in the literature we will be using.

Theorem 8.1 (Delegation Principle for Common Agency Games in [20])**.** *For each σ^* perfect Bayesian equilibrium in the game in mechanisms which message space is $\mathcal{M}$, there exists a perfect Bayesian equilibrium $\tilde{\sigma}^*$ in the game of menus of at most $|\mathcal{M}|$ elements that induces the same distribution on actions.*

Theorem 8.2 (Theorem 6 in [25])**.** *In the common agency problem with a single agent, let (c^*, δ) be an equilibrium relative to the set of menus Γ^M. Then (c^*, δ) is weakly robust.*

Below are the proofs of the two simplifying theorems.

Proof of theorem 5.1. From theorem 1 in [20] (theorem 8.1), we know that any equilibrium relative to a feasible set of mechanisms with communication space $\mathcal{M}$ can be replicated by an equilibrium relative to the set of menus which image space has cardinality of at most the cardinality of $\mathcal{M}_i$ for all designers $i = 1, \dots, n$, i.e., there exists an equilibrium relative to the set of menus of cardinality less than that of $\mathcal{M}_i$ that generates the same simple actions for all players and preserves all payoffs. This is the delegation principle. Applied to the universal communication space defined in [12], it means that all equilibria relative to the universal set of mechanisms Γ^u can be replicated by an equilibrium in the set of menus Γ^M that preserves all payoffs. So we only have to show that, for any equilibrium relative to the set of menus, there exists an equilibrium in incentive compatible direct mechanisms that preserves all payoffs.

Let $(\delta_1^*, \dots, \delta_n^*, c^*) \in \prod_{i=1}^n \Delta(\Gamma_i^M) \times \Delta(\mathcal{E}) \times \Delta(A)$ be an equilibrium relative to the set of menus Γ^M. $c^*(\gamma, t) = (\tilde{m}^c(\gamma, t), \tilde{a}^c(\mu, t)$ is the continuation equilibrium played by the common receiver of type $t \in T$ when she is offered γ. Then her expected payoff is

$$u_R^{c^*}(\gamma, t) = \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \dots \int_{\Delta(\Omega_n)} \int_{\Omega} \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(m_n)(\mu_n) \dots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c(m)$$

where γ_i is the menu drawn from the randomization δ_i^*. Then γ_i maps a message $m_i \in \mathcal{E}_i$ into the set of simple action: $\gamma_i(m_i) = \tau_i(\mu)$.

Let $\tau^{c^*}(t)$ be the distribution on $\prod_{i=1}^n \Delta(\Delta(\Omega))$ associated with the continuation equilibrium c^* conditional on receiver's type $t \in T$. These distributions completely characterizes the receiver decision strategy since the latter is a deterministic function of beliefs on ω. Then $\tau^{c^*}(t) = \delta^*(\gamma)\tilde{m}^c(\gamma, t)$, where $\delta^* \in \Delta(\Gamma)$ is the joint distribution on simple actions played in the equilibrium of the menu game.

Since c^* is a continuation equilibrium, the receiver does not have any profitable deviation and $\forall \gamma \in \text{supp } \delta^*$, $\forall i = 1, \ldots, n$, $\forall \tau_i \in \text{supp } \tilde{m}_i^c(\gamma, t), \forall \tau_i' \in \mathcal{M}_i^{\gamma_i}$[29]

$$u_R^{c^*}(\gamma, t) \equiv u_R^{(\tau_i^*, \tau_{-i}^*)}(\gamma, t) =$$
$$\int_{\mathcal{M}_{-i}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_{\Omega} \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(\tau_n^*)(\mu_n) \ldots d\tau_i^*(\mu_i) \ldots d\gamma_1(\tau_1^*)(\mu_1) d\tilde{m}_{-i}^c(\tau_{-i}^*)$$
$$\geq u_R^{c'}(\gamma, t) \equiv u_R^{(\tau_i', \tau_{-i}^*)}(\gamma, t) =$$
$$\int_{\mathcal{M}_{-i}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_{\Omega} \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(\tau_n^*)(\mu_n) \ldots d\tau_i'(\mu_i) \ldots d\gamma_1(\tau_1^*)(\mu_1) d\tilde{m}_{-i}^c(\tau_{-i}^*)$$

where $u_R^{(\tau_i, \tau_{-i})}(\gamma, t)$ is a new notation to insist on the reports made in equilibrium and τ_i^* is the simple action taken by sender D_i in the continuation equilibrium c^*. Note that the communication equilibrium is well defined since the message space is a closed subset $\mathcal{E}$, which is a compact metric space. Hence for all closed subset $U_i \subset \mathcal{E}_i \subset \Delta(\Delta(\Omega_i))$, there exists τ_i such that

$$u_R^{(\tau_i, \tau_{-i})}(\gamma, t) \geq u_R^{(\tau_i', \tau_{-i})}(\gamma, t)$$

for all $\tau_{-i} \in \mathcal{E}_{-i}$, all $\tau_i' \in U$, and all $t \in T$.

From here, we can construct an equilibrium in incentive compatible direct persuasion mechanisms that preserves all payoffs. For all $t \in T$, let $\tau_i^*(t)$ be the simple action that generates the same beliefs on ω_i as the randomization on simple actions induced by sender D_i's randomization on menus and the receiver's communication strategy in equilibrium. Such an action $\tau_i^*(t)$ exists by lemma 3.1. Note also that $\tau_i^*(t)$ verifies the above no deviation inequality.

Let $\tilde{a}^*(t)$ be the distribution on actions induced by c^* for a receiver of type t, $\tilde{a}^*(t) = \tau^{c^*}(t)(\mu)a^c(\mu, t)$.

We define now the incentive compatible direct persuasion mechanisms that will constitute the new

[29] $\mathcal{M}_i^{\gamma_i}$ is the message space associated with menu γ_i. In standard notation, x_{-i} corresponds to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

equilibrium in direct persuasion mechanisms. For all $i = 1, \ldots, n$, define the contract d_i^{DM} as

$$\begin{aligned} d_i^{DM}: \quad & T \to \mathcal{E}_i \\ & t \to \tau_i^*(t) \end{aligned}$$

where $\tau_i^*(t)$ is defined above. We only have left to show that this direct mechanisms are incentive compatible, that the truth-telling continuation strategy associated with $(d_1^{DM}, \ldots, d_n^{DM})$ is optimal, and that $(d_1^{DM}, \ldots, d_n^{DM})$ is an equilibrium of the senders game relative to the set of direct persuasion mechanisms.

Suppose that the common receiver responds to the array of offers $(d_1^{DM}, \ldots, d_n^{DM})$ by reporting her true type and by taking the action strategy $\tilde{a}^c(\mu, t)$. Then sender D_i's ex-ante expected payoff is

$$\begin{aligned} \bar{v}_i(d_1^{DM}, \ldots, d_n^{DM}) &= \int_T \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^c(\mu, t)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \ldots d\tau_1^*(t)(\mu_1)\, dF(t) \\ &= \int_T \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^*(t)\, d\mu_0(\omega)\, dF(t) \end{aligned}$$

where the equality comes from Bayes-plausibility, the independence of the ω_i's, and the definition of $\tilde{a}^*(t)$. Of course, sender D_i's payoff is unchanged since the receiver's decision strategy remains the same and the mechanisms are payoff irrelevant.

For the receiver's payoff, observe that, in the menu equilibrium,

$$\begin{aligned} u_R^{c^*}(t) &= \int_{\Gamma^M} \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_\Omega \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(m_n)(\mu_n) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c(m)\, d\delta^*(\gamma) \\ &= \int_{\Gamma^M_{-n}} \int_{\mathcal{M}_{-n}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_{n-1})} \int_{\Gamma^M_n} \int_{\mathcal{M}_\backslash} \int_{\Delta(\Omega_n)} \int_\Omega \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega) \\ & d\gamma_n(m_n)(\mu_n)\, d\tilde{m}_n^c(m_n)\, d\delta_n^*(\gamma_n)\, d\gamma_{n-1}(m_{n-1})(\mu_{n-1}) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c_{-n}(m_{-n}) d\delta^*_{-n}(\gamma_{-n}) \\ &= \int_{\Gamma^M_{-n}} \int_{\mathcal{M}_{-n}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_{n-1})} \int_{\Delta(\Omega_n)} \int_\Omega \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \\ & d\gamma_{n-1}(m_{n-1})(\mu_{n-1}) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c_{-n}(m_{-n}) \end{aligned}$$

where the first equality is Fubini's theorem and the second comes from the definition of τ_n^*. Repeating this operation n times yields

$$u_R^{c^*}(t) = \int_{\Delta(\Omega_1)} \dots \int_{\Delta(\Omega_{n-1})} \int_{\Delta(\Omega_n)} \int_{\Omega} \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \dots d\tau_1^*(t)(\mu_1)$$

Furthermore, recall that $\tau_i^*(t)$ verifies the no deviation inequality, and that any message played in the support of the original mixed communication equilibrium must yield the same expected payoff. Thus the common receiver payoff's is also preserved.

We still have to show that the array $(d_1^{DM}, \dots, d_n^{DM}, m^T, a^c(\mu, t))$, where m^T is the truth-telling communication strategy, is an equilibrium in the multi-senders common-receiver game relative to the set of direct persuasion mechanisms.

We first show that for all t, $a^c(\mu, t)$ is a best response. This is immediate as $a^c(\mu, t)$ is part of the original continuation equilibrium for the menu game. Any deviation available in the new game in direct mechanisms was already available before. So no deviation to another decision strategy can be profitable to the common receiver.

Secondly, we show that reporting the truth about her type is an optimal communication strategy for the receiver. Again $\tau_i^*(t)$ verifies the no deviation inequality and is payoff equivalent to the simple action for sender D_i taken in the menu equilibrium. However menus being independent of types and $\tilde{m}^c$ being played in the continuation equilibrium, it implies that telling the truth is optimal.

Hence $\forall t \in T$ and $\forall i = 1, \dots, n$, $\tau_i^*(t)$ is payoff equivalent to the maximizing communication strategy $\delta_{\tau_i}(t)$ associated with δ_i^* for the receiver of type t. Furthermore $\forall t' \in T$, $\tau_i^*(t')$ was constructed such that it is payoff equivalent to a communication strategy available to R in the menu game. Then to report $t' \neq t$ is not a profitable deviation. If it were, there would be a contract $\tau_i^*(t')$ and therefore a communication strategy $\delta_{\tau_i}(t')$ available to receiver of type t in the menu game such that it would yield an higher payoff than $\delta_{\tau_i}(t)$ in the menu game. But then $\delta_{\tau_i}(t)$ cannot be part of the communication strategy of the receiver in equilibrium. This is a contradiction. Since a payoff equivalent deviation was feasible in the original equilibrium and such deviation was not profitable, it cannot be profitable in the new continuation equilibrium.

So we showed that reporting her true type and selecting decision strategy $\tilde{a}^c(\mu, t)$ is a continuation equilibrium for the receiver when $(d_1^{DM}, \dots, d_n^{DM})$ are offered. It remains to show that no information designer D_i has an incentive to unilaterally deviate in Γ_i^{DM} given this continuation equilibrium. We show that it holds by showing that any deviation in direct persuasion mechanisms is payoff equivalent to a deviation in menus in the original game. Such deviation was not profitable before and the equilibrium payoffs are preserved, so the new deviation is unprofitable too.

Suppose that sender D_i deviates to a strategy[30] $\tau_i' \in \Gamma_i^{DM}$ and that all other senders D_{-i} still play d_{-i}^{DM}. The agent continuation strategy is composed of the truth-telling communication strategy m^T and of the decision strategy $\tilde{a}^c(\mu, t)$. The payoff of the deviator is

$$\begin{aligned}\bar{v}_i(d_1^{DM}, \ldots, \delta_i', \ldots, d_n^{DM}) &= \int_T \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^c(\mu, t)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \ldots, d\tau_i'(t)(\mu_i) \\ &\quad \ldots d\tau_1^*(t)(\mu_1)\, dF(t) \\ &= \int_{\Gamma^M} \int_{\mathcal{M}} \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_\Omega \int_A u_R(a, \omega, t)\, d\tilde{a}^c(\mu, t)(a)\, d\mu(\omega)\, d\gamma_n(m_n)(\mu_n) \\ &\quad \ldots d\tau_i'(t)(\mu_i) \ldots d\gamma_1(m_1)(\mu_1) d\tilde{m}^c(m)\, d\delta^*(\gamma)\end{aligned}$$

which shows that the payoff sender D_i gets by deviating is the same as the payoff he would have got when deviating to τ_i' in the menu game. The latter deviation was indeed feasible since the set of direct persuasion mechanisms is included in the set of menus. Furthermore, it was not a profitable deviation before, so it must still be unprofitable. Thus no designer has an incentive to deviate. This concludes the proof. □

Proof of theorem 5.2. The existence of a pure strategy payoff equivalent equilibrium is a consequence of lemma 3.1. Furthermore, from Peters' theorem 6 in [25], we know that any equilibrium relative to the set of menus Γ^M is weakly robust. So we only have to show that any pure strategy equilibrium in incentive compatible direct persuasion mechanism is also an equilibrium relative to the set of menus.

Let $(\gamma_1^*, \ldots, \gamma_n^*, c^*)$ be an equilibrium of the multi-senders common-receiver game relative to the set of incentive compatible direct persuasion mechanisms. Since each γ_i^* is incentive compatible, the common receiver reports truthfully, which is a pure strategy. It follows that for all $t \in T$, the receiver R chooses a single contract from γ_i for all $i = 1, \ldots, n$. Denote $\tau_i^*(t)$ the simple action chosen by the common receiver of type $t \in T$ from information designer D_i.

Note that D_i's expected payoff is given by

$$\begin{aligned}\bar{v}_i(d_1^{DM}, \ldots, d_n^{DM}) &= \int_T \int_{\Delta(\Omega_1)} \cdots \int_{\Delta(\Omega_n)} \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^c(\mu, t)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \ldots d\tau_1^*(t)(\mu_1)\, dF(t) \\ &= \int_T \int_\Omega \int_A v_i(a, \omega, t)\, d\tilde{a}^*(t)\, d\mu_0(\omega)\, dF(t)\end{aligned}$$

[30]Considering only deviations in pure strategies is without loss of generality from lemma 3.1.

where $\tilde{a}^*(t)$ is the distribution on actions induced in equilibrium: $\tilde{a}^*(t) = \tau_1^*(t)(\mu_1)\dots\tau_n^*(t)(\mu_n)a^c(\mu,t)$.

The problem is to extend the continuation equilibrium $c^* = (m^T, \tilde{a}^c(\mu,t))$, where m^T is the truthful reporting communication strategy, such that it preserves the initial equilibrium in the menu space.

Define c^M, the extended continuation equilibrium on the set of menus Γ^M. Note that the common receiver should respond to any mechanism $\gamma \in \Gamma^{DM} \subset \Gamma^M$ in exactly the same way as he does in the original equilibrium. Then, for all $\gamma \in \Gamma^{DM}$, define

$$\forall t \in T, \quad c^M(\gamma, t) = c^*(\gamma, t)$$

Since $\Gamma^{DM} \subset \Gamma^M$, the equilibrium offers $(\gamma_1^*, \dots, \gamma_n^*)$ are also in Γ^M, and the equilibrium payoffs are preserved. Furthermore c^M is a best response in this case as no new deviation is available to the receiver.

Consider now a particular information designer D_i. We want to show that there exists a continuation equilibrium such that this sender has no profitable deviation in Γ^M. We do so by construction, i.e., we define a continuation equilibrium that yields a lower payoff than the equilibrium's payoff for any deviation of information designer D_i.

Suppose that D_i deviates and offer a mechanism $\gamma_i' \in \Gamma^M \setminus \Gamma^{DM}$, while every other sender D_{-i} still offer their equilibrium mechanism in Γ^{DM}. We construct the continuation equilibrium c^M such that the deviation to γ_i' has the same consequence as a deviation in Γ^{DM}. For each type $t \in T$, choose an array of incentive compatible direct persuasion mechanisms

$$\{\tau_1^*(t), \dots, \tau_n^*(t)\} \subset \{\gamma_1^*, \dots, \gamma_i', \dots, \gamma_n^*\}$$

where $\tau_j^*(t) \in Im(\gamma_j^*)$, $j \neq i$ and $\tau_i'(t) \in Im(\gamma_i')$ such that

$$\tau_i'(t) \in \arg\max_{\tau_i \in \gamma_i'} \int_{\Delta(\Omega_1)} \dots \int_{\Delta(\Omega_{n-1})} \int_{\Delta(\Omega_n)} \int_{\Omega} \int_A u_R(a,\omega,t)\, d\tilde{a}^c(\mu,t)(a)\, d\mu(\omega)\, d\tau_n^*(t)(\mu_n) \dots d\tau_i(t)(\mu_i) \dots d\tau_1^*(t)(\mu_1)$$

Define the direct mechanism $d_i' = \{\tau_i'(t)\}_{t \in T}$ for the deviating sender. The choice set offered to the receiver when mechanisms are $(\gamma_1^*, \dots, \gamma_i', \dots \gamma_n^*)$ contains the choice set available to the receiver when the senders play mechanisms $(\gamma_1^*, \dots, d_i', \dots \gamma_n^*)$, since $Im(d_i') \subset Im(\gamma_i')$. Then, for each type $t \in T$ and each γ_i', $c^M(\gamma_1^*, \dots, \gamma_i', \dots \gamma_n^*) = (m_{-i}^T, \tau_i'(t), \tilde{a}^c(\mu,t))$ is a continuation equilibrium (by construction). The distributions on actions induced by c^M are identical when $(\gamma_1^*, \dots, \gamma_i', \dots \gamma_n^*)$ and

$(\gamma_1^*, \ldots, d_i', \ldots \gamma_n^*)$ are offered. Then the payoff of all senders are also unchanged. In particular the payoff of information designer D_i is the same when deviating to offer γ_i' and d_i'. But d_i' was a feasible deviation of the original game in direct persuasion mechanisms. So the payoff for designer D_i associated with d_i' is less than the payoff associated with γ_i^* and therefore γ_i' is not a profitable deviation.

We have shown that there exists a continuation equilibrium such that no deviation in Γ^M is profitable. This concludes the proof. $\square$

8.1.2 Proofs of the two existence criteria

We start with a few definitions and propositions from [28] that will be useful.

Definition 4. *Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $\tau \in \mathcal{E}$ if there exists $\bar{\tau}_i \in \mathcal{E}_i$, such that $v_i(\bar{\tau}_i, \tau'_{-i}) \geq \alpha$ for all τ'_{-i} in some open neighbor of τ_{-i}.*

A game is payoff secure when all players can secure a payoff at all points of the action space.

Definition 5. *A game $G = (\mathcal{E}_i, v_i)_{i=1}^n$ is reciprocally upper semicontinuous if whenever (τ, v) is in the closure of the graph of its vector payoff function and $v_i(\tau) \leq v_i$ for every player i, then $v_i(\tau) = v_i$ for every player i.*

Definition 6. *Let $G = (\mathcal{E}_i, v_i)_{i=1}^n$ be a game. A mixed extension G_m of G is defined by $G_m = \left(\Delta(\mathcal{E}_i), \int_{\mathcal{E}} v_i \, d\tau\right)_{i=1}^n$ for all $\tau \in \prod_{i=1}^n \Delta(\mathcal{E}_i)$.*

We will also use Proposition 5.1 in [28].

Proposition 8.3 (Proposition 5.1 in [28])**.** *If $\sum_{i=1}^n v_i(\tau)$ is upper semicontinuous in τ on $\mathcal{E}$, then $\sum_{i=1}^n \int_{\mathcal{E}} v_i(\tau) \, d\delta(\tau)$ is upper semi continuous in δ on $\prod_{i=1}^n \Delta(\mathcal{E}_i)$. Consequently, the mixed extension of the game is reciprocally upper semicontinuous.*

Finally, corollary 5.2 in [28] will be the main ingredient of our proof, so we reproduce it here.

Theorem 8.4 (Corollary 5.2 in [28])**.** *Suppose that $G = (\mathcal{E}_i, v_i)_{i=1}^n$ is a compact Hausdorff game. Then G possesses a mixed strategy Nash equilibrium if its mixed extension, G_m, is better-reply secure. Moreover G_m is better-reply secure if it is both reciprocally upper semicontinuous and payoff secure.*

Before presenting the proofs, we also state a lemma, which is going to be useful in the proof of the two existence criteria.

Lemma 8.5. *The continuation equilibrium is well defined in the game relative to the incentive compatible direct persuasion mechanisms.*

Proof. From the continuation equilibrium subsection 4.1, we only have to verify that there exists an optimal communication strategy for the common receiver R. This is the case when the message space $\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i$ is compact. When designers compete in incentive compatible direct persuasion mechanisms, they offer mechanisms, which message space, $\mathcal{M}_i$, is a closed subset of T for all $i = 1, \ldots, n$. Since we assumed that T is a compact metric space, each $\mathcal{M}_i$ is compact, and so is $\mathcal{M}$. $\square$

Proof of theorem 5.4. The receiver has no private information of her own, $|T| = 1$. Then an incentive compatible direct persuasion mechanism for D_i, which is a T-measurable mapping from T to $\mathcal{E}_i$, corresponds to a single simple action $\tau_i \in \mathcal{E}_i$. Any strategy δ_i played by sender D_i is then a distribution on simple actions, $\delta_i \in \Delta(\mathcal{E}_i)$.

Furthermore from lemma 8.5, a continuation equilibrium exists in the game relative to the incentive compatible direct persuasion mechanisms. When the receiver has no private information of her own, this continuation equilibrium is fully summarized by $a^c(\mu)$, the decision strategy of the receiver. Therefore to show the existence of an equilibrium of the multi-senders common-receiver game, we only have to show that the normal form game between senders, defined by the continuation equilibrium of the common receiver, where senders feasible actions are $\mathcal{E}_i$, has a Nash equilibrium.

Let $c^* = (m^T, \tilde{a}^c(\mu))$ be the continuation equilibrium played by the receiver. To show the existence of an equilibrium in the game G^{c^*} induced by c^* when the payoffs of the senders are secure, we use Reny's corollary 5.2 in [28] (theorem 8.4). Hence it tells us that a mixed strategy Nash equilibrium of G^{c^*} exists if G^{c^*} is a compact Hausdorff game and if its mixed extension $G_m^{c^*}$ is payoff secure and reciprocally upper semicontinuous.

We show that G^{c^*} is a compact Hausdorff game, that is, each $\mathcal{E}_i$ is compact Hausdorff.

Recall that $\forall i = 1, \ldots, n$, $\mathcal{E}_i \subset \Delta(\Delta(\Omega_i))$ is the set of Bayes-plausible distribution on posterior beliefs. Since Ω_i is compact metric, so is $\Delta(\Omega_i)$, the set of Borel probabilities on Ω_i. Hence $\Delta(\Omega_i)$ is compact in the weak* topology as a consequence of Riesz representation theorem and Alaoglu's theorem. See for example proposition 5.3 in [31], and note that the convergence in the Lévy–Prokhorov metric is the same as weak convergence when the underlying space is separable. Reproducing the same reasoning, $\Delta(\Delta(\Omega_i))$ is a compact metric space. Furthermore all metric spaces are Hausdorff. Then

$\Delta(\Delta(\Omega_i))$ is compact Hausdorff. So to show that $\mathcal{E}_i$ is compact Hausdorff, we only have to show that it is closed. This is obvious. Let $(\tau_i^n)_{n\in\mathbb{N}} \subset \mathcal{E}_i$ be a convergent sequence and call its limit τ_i. Then $\forall n \in \mathbb{N},\ \int_{\Delta(\Omega_i)} \mu \, d\tau_i^n(\mu) = \mu_0^i$. But is $\int_{\Delta(\Omega_i)} \mu \, d\tau_i^n(\mu)$ is continuous in τ_i^n. Hence $\Delta(\Omega_i)$ is compact, the identity function is continuous, so the random variable μ is dominated, and the continuity comes from Lebesgue dominated convergence theorem. Therefore we can take the limit in the above expression and $\int_{\Delta(\Omega_i)} \mu \, d\tau_i(\mu) = \mu_0^i$.

Thus every $\mathcal{E}_i$ is compact Hausdorff, and G^{c^*} is a compact Hausdorff game.

We show that the game is reciprocally upper semicontinuous. The payoffs of the senders in G^{c^*} are given, for any $\tau \in \mathcal{E} = \prod_{i=1}^n \mathcal{E}_i$, by

$$\bar{v}_i(\tau) = \int_{\Delta(\Omega)} \int_\Omega \int_A v_i(a,\omega) \, d\tilde{a}^c(\mu)(a) \, d\mu(\omega) \, d\tau(\mu)$$

where τ is the distribution on $\Delta(\Omega)$ generated by the actions of the senders. Define $\hat{v}_i(\mu)$ as

$$\hat{v}_i(\mu) = \int_\Omega \int_A v_i(a,\omega) \, d\tilde{a}^c(\mu)(a) \, d\mu(\omega)$$

We first want to show that $\sum_{i=1}^n \hat{v}_i(\mu)$ is upper semicontinuous. Note that

$$\sum_{i=1}^n \hat{v}_i(\mu) = \sum_{i=1}^n \max_{\tilde{a}\in\tilde{a}^*(\mu)} \int_\Omega \int_A v_i(a,\omega) \, d\tilde{a}(a) \, d\mu(\omega)$$

since $\tilde{a}^c$ is the continuation strategy played in equilibrium. Furthermore, given $\tilde{a} \in \Delta(A)$, the random variable $\int_A v_i(a,\omega) \, d\tilde{a}(a)$ is dominated by the constant random variable $\max_{\omega\in\Omega} \max_{a\in A} v_i(a,\omega)$ (where the maximum exists by Heine's theorem). Then, by Lebesgue dominated convergence theorem, $\int_\Omega \int_A v_i(a,\omega,t) \, d\tilde{a}(a) \, d\mu(\omega)$ is continuous in μ for any given $\tilde{a} \in \Delta(A)$. Therefore $\sum_{i=1}^n \int_\Omega \int_A v_i(a,\omega) \, d\tilde{a}(a) \, d\mu(\omega)$ is continuous in μ for any given $\tilde{a} \in \Delta(A)$.

Thus, from Berge maximum theorem, $\sum_{i=1}^n \hat{v}_i(\mu)$ is upper semicontinuous if $\tilde{a}^*(\mu)$ is an upper hemicontinuous, compact valued, non empty correspondence from $\Delta(\Omega_i)$ into $\Delta(A)$. $\tilde{a}^*(\mu)$ solves the maximization program of the receiver in the second stage of the continuation problem:

$$\tilde{a}^*(\mu) = \arg \max_{\tilde{a}\in\Delta(A)} \int_\Omega \int_A u_R(a,\omega) \, d\tilde{a} \, d\mu(\omega)$$

From subsection 4.1, we know that $\tilde{a}^*(\mu)$ is an upper hemicontinuous, compact valued, non empty correspondence from $\Delta(\Omega_i)$ into $\Delta(A)$. Then $\sum_{i=1}^n \hat{v}_i(\mu)$ is upper semicontinuous in μ.

Secondly we show that $\sum_{i=1}^n \bar{v}_i(\tau) = \sum_{i=1}^n \int_{\Delta(\Omega)} \hat{v}_i(\mu) d\tau(\mu)$ is upper semicontinuous in τ on $\mathcal{E}$. This follows from proposition 5.1. and its proof in Reny [28] (proposition 8.3).

Finally, applying again proposition 5.1. in Reny [28] to $\sum_{i=1}^n \int_{\Delta(\Omega)} \hat{v}_i(\mu) d\tau(\mu)$, we have that

$$\sum_{i=1}^{n} \int_{\Delta(\Omega)} \bar{v}_i(\tau) d\delta(\tau)$$

is upper semicontinuous in δ on $\prod_{i=1}^n \Delta(\mathcal{E}_i)$ and that the extension of the sender game in mixed strategy, $G_m^{c^*}$, is reciprocally upper semicontinuous.

Finally $G_m^{c^*}$ is payoff secure from lemma 3.1 since we assumed that G^{c^*} is. Thus, by Reny's corollary 5.2, a mixed strategy Nash equilibrium of G^{c^*} exists. By lemma 3.1 again, a payoff equivalent pure strategy equilibrium of the normal form game among sender defined by c^* exists. This concludes the proof. □

Proof of theorem 5.6. We want to show that there exists an equilibrium of the multi-senders common-receiver game relative to the set of incentive compatible direct mechanisms.

From lemma 8.5, a continuation equilibrium exists in the game relative to the direct persuasion mechanisms. Let $c^* = (\tilde{m}^c, a^c(\mu, t))$ be the continuation equilibrium. Therefore to show the existence of an equilibrium of the multi-senders common-receiver game, we only have to show that the normal form game between senders, defined by the continuation equilibrium of the common receiver c^*, has a Nash equilibrium. Call this game G. The set of actions for each sender i in G is the set of truthful direct persuasion mechanisms, Γ_i^{ICDM}.

Recall that a direct persuasion mechanism for D_i, $\gamma_i \in \Gamma_i^{DM}$, is characterized by a measurable mapping

$$\begin{aligned} \gamma_i : T &\to \mathcal{E}_i \\ t &\to \gamma_i(t) \end{aligned}$$

An incentive compatible direct persuasion mechanism is a direct persuasion mechanism that induces truth-telling as an optimal communication strategy. A direct persuasion mechanism is incentive compatible if and only if for all possible type $t \in T$, the common receiver R has an incentive to report her type t truthfully.

To show the existence of an equilibrium in the game G induced by c^* when the payoffs of the senders are secure, we use Reny's corollary 5.2 in [28] (theorem 8.4). Hence it tells us that a mixed strategy Nash equilibrium of G exists if G is a compact Hausdorff game and if its mixed extension G_m is payoff secure and reciprocally upper semicontinuous.

We first show that the set of direct persuasion mechanisms is compact, and therefore that the set of incentive compatible direct persuasion mechanism is also. Then we show that the game is reciprocally upper semicontinuous. Finally we apply Reny's corollary 5.2.

We show that G is a compact Hausdorff game, that is, each Γ_i^{ICDM} is compact Hausdorff. We begin by showing that Γ_i^{DM} is compact Hausdorff.

$\forall i = 1, \ldots, n$, $\mathcal{E}_i \subset \Delta(\Delta(\Omega_i))$, is compact metric. This comes from the proof of theorem 5.4. It is metrized with the metric defined in [29]. Then, from [29], there exists a reproducing kernel Hilbert space embedding for $\Delta(\Delta(\Omega_i))$. Then $\Delta(\Delta(\Omega_i))$ is a Hilbert space.[31] From Fraňková-Helly selection theorem, it follows that Γ_i^{DM} is compact and metrizable. Hence all elements γ_i of Γ_i are indeed regulated functions since both image space $\mathcal{E}_i$ and the preimage space T are compact metric space. Finally we can equip Γ_i^{DM} with a metric to make it a compact metric space. Therefore Γ_i^{DM} is Hausdorff as a metrizable space.

Thus every Γ_i^{DM} is compact Hausdorff. We show that it implies that Γ_i^{ICDM} is compact Hausdorff. This is the case if Γ_i^{ICDM} is a closed subset of Γ_i^{DM}. So we have to show that $\Gamma_i^{ICDM} \subset \Gamma_i^{DM}$ is indeed closed. Fix $\tau_{-i} \in \Gamma_{-i}^{ICDM}$.

The set of incentive compatible direct persuasion mechanisms available to information designer D_i is characterized by

$$\Gamma_i^{ICDM} = \{\tau_i : T \to \mathcal{E}_i \text{ such that } U_R^{\tau_i}(t,t) \geq U_R^{\tau_i}(t,t') \,\forall t,\, t' \in T\}$$

where $U_R^{\tau_i}(t,t') = \int_{\Delta(\Omega_{-i})} \int_{\Delta(\Omega_i)} \int_\Omega \int_A u_R(a,\omega,t)\, d\tilde{a}^c(\mu,t)(a)\, d\mu(\omega)\, d\tau_i(t')(\mu_i) d\tau_{-i}(t)$.

[31] What is important here is the existence of a Hilbert structure compatible with the weak* topology for the space of Borel probabilities, so we can apply Fraňková-Helly selection theorem. I think that it is guaranteed by the Hilbert embedding described in [29], but I am not completely sure. Note however that, if Ω is finite, every $\mathcal{E}_i$ can be assimilated to a closed subset of an Euclidean space, so there is no problem.

Let $(\tau_i^n)_{n\in\mathbb{N}} \subset \Gamma_i^{ICDM}$ be a sequence of incentive compatible direct persuasion mechanisms. Suppose that $(\tau_i^n)_{n\in\mathbb{N}}$ converges to τ_i. We want to show that the convergence occurs in Γ_i^{ICDM}, i.e., that $\tau_i \in \Gamma_i^{ICDM}$. $\forall n \in \mathbb{N}$, $U_R^{\tau_i^n}(t,t) \geq U_R^{\tau_i^n}(t,t') \forall t,\, t' \in T$. Furthermore $U_R^{\tau_i^n}(t,t')$ is continuous in τ_i^n from Lebesgue dominated convergence theorem and the linearity of the integral. Then we can take the limit in the inequality and we obtain $U_R^{\tau_i}(t,t) \geq U_R^{\tau_i}(t,t')$, that is τ_i is incentive compatible.

Then Γ_i^{ICDM} is closed in Γ_i^{DM}. Therefore it is compact Hausdorff. This result is similar to Myerson's theorem 1 in [21], although the proof differs. Then G is a compact Hausdorff game.

We show that the game is reciprocally upper semicontinuous. The proof follows the same step as the proof of reciprocal upper semicontinuity of the game $G_m^{c^*}$ in the proof of theorem 5.4.

The payoffs of the senders in G^{c^*} are given, for any $\tau \in \mathcal{E} = \prod_{i=1}^n \mathcal{E}_i$, by

$$\bar{v}_i(\tau) = \int_T \int_{\Delta(\Omega)} \int_\Omega \int_A v_i(a,\omega,t)\, d\tilde{a}^c(\mu,t)(a)\, d\mu(\omega)\, d\tau(t)(\mu)\, dF(t)$$

where τ is the distribution on $\Delta(\Omega)$ generated by the actions of the senders.

Reproducing the proof of theorem 5.4, we have that, for all $t \in T$,

$$\sum_{i=1}^n \int_{\Delta(\Omega)} \int_\Omega \int_A v_i(a,\omega,t)\, d\tilde{a}^c(\mu,t)(a)\, d\mu(\omega)\, d\tau(t)(\mu)$$

is upper semicontinuous in $\tau(t)$ on $\mathcal{E}(t) \subset \mathcal{E}$, where $\mathcal{E}(t)$ is the set of incentive compatible direct persuasion mechanisms chosen by the common receiver R of type t. This implies that the sum of the expected payoffs is upper semicontinuous and, by proposition 5.1. in [28] (proposition 8.3), that G_m is reciprocally upper semicontinuous.

Finally G_m is payoff secure from lemma 3.1 since we assumed that G is. Thus, by Reny's corollary 5.2, a mixed strategy Nash equilibrium of G exists. By lemma 3.1 again, an equivalent pure strategy equilibrium of the normal form game among sender defined by c^* exists. This concludes the proof. $\square$

8.2 Appendix for section 6

Proof of proposition 6.1. When $\mu_0 \geq \frac{\kappa}{1+\kappa}$, the agency already takes firm's preferred action, so no revelation is optimal.

Suppose now that $\mu_0 < \frac{\kappa}{1+\kappa}$. Finding the equilibrium is an easy application of Bayesian persuasion.

Note that $\hat{v}_j(a, \mu_j, \mu_{-j})$ takes value 0 when $a_j = 0$ and value > 0 when $a_j = 1$, and a_j is the only action player j has some control over. So it is optimal for the firm (the sender) to maximize the probability that the agency (the receiver) takes action $a_j = 1$.

From Kamenica and Gentzkow [16], we know that the optimal strategy of both firms (when they are alone) is to use a binary signal, and that one signal must induce beliefs $\mu_j'' = 0$. So we only need to find the remaining two parameters: λ_j and μ_j'. From Bayes-plausibility,

$$\lambda_j \mu_j' + (1 - \lambda_j)\mu_j'' = \mu_0$$

Then $\lambda_j = \frac{\mu_0}{\mu_j'}$. So the problem of firm j is to pick λ_j that maximizes her expected utility

$$\mathbb{E}_{\tau_j, \tau_{-j}}[\hat{v}_j(\mu_j, \mu_{-j})] = p\left[\lambda_j(1 - \lambda_{-j}) + \frac{1}{2}\lambda_j\lambda_{-j}\right]\mathbf{1}_{\{\frac{\mu_0(1+\kappa)}{\kappa} \geq \lambda_j\}}$$

This is maximized by $\lambda_j = \frac{\mu_0(1+\kappa)}{\kappa}$ and then $\mu_j' = \frac{1+\kappa}{\kappa}$. Then firm j's expected payoff is

$$\mathbb{E}_{\tau_j, \tau_{-j}}[\hat{v}_j(\mu_j, \mu_{-j})] = p\frac{\mu_0(1+\kappa)}{\kappa}\left[1 - \frac{1}{2}\frac{\mu_0(1+\kappa)}{\kappa}\right]$$

We still need to show that this is an equilibrium and that it is unique. By unique, we mean that there is no other distribution on beliefs that constitutes a Bayesian Nash equilibrium of the game.

That it is an equilibrium is obvious, as the strategy found above for firm j is a dominant strategy. Hence increasing μ_j' or μ_j'' reduces λ_j by Bayes-plausibility and reduces firm j's payoff too. Decreasing μ_j' yield payoffs zero.

No other distribution on beliefs can be induced in equilibrium. Suppose that there exists another equilibrium $(\tau_2', \tau_2') \neq (\tau_1^*, \tau_2^*)$. Then $\exists j \in \{1, 2\}$ such that $\tau_j' \neq \tau_j^*$. But deviating to τ_j^* is then profitable since it is a dominant strategy, and (τ_2', τ_2') cannot be an equilibrium. □

Proof of proposition 6.6. Note that, by lemma 6.4 and the first indifference condition, $p_{0,j} = p_{0,-j} \equiv p_0$. Therefore $G_j(\hat{\mu}) = G_{-j}(\hat{\mu}) \equiv G(\hat{\mu})$ and $\forall \mu \in [\frac{\kappa}{1+\kappa})$, $G_j(\mu) = G_{-j}(\mu) \equiv G(\mu)$. Finally, from lemma 6.4 and the second indifference conditions, it follows that $p_{1,j} = p_{1,-j} \equiv p_1$. All equilibria of the game are symmetric.

We now solve for the equilibrium strategies. Distinguish three cases. First suppose that $p_0 + G(\hat{\mu}) =$

1, i.e., $p_1 = 0$. Then the equilibrium strategies solve

$$\begin{cases} \mu_0 = (1-p_0)\frac{\frac{\kappa}{1+\kappa}+\hat{\mu}}{2} \\ \hat{\mu} = \frac{\kappa}{(1+\kappa)p_0} \\ p_0 + G(\hat{\mu}) = 1 \end{cases}$$

This is a system of three equations in three unknowns. It has a unique solution (since $p_0 \in (0,1)$). Hence

$$\begin{aligned} p_0 &= \frac{\kappa+1}{\kappa}\left(\sqrt{\mu_0^2 + \frac{\kappa^2}{(1+\kappa)^2}} - \mu_0\right) \\ G(\hat{\mu}) &= 1 - \frac{\kappa+1}{\kappa}\left(\sqrt{\mu_0^2 + \frac{\kappa^2}{(1+\kappa)^2}} - \mu_0\right) \\ \hat{\mu} &= \left(\frac{\kappa+1}{\kappa}\right)^2 \left(\sqrt{\mu_0^2 + \frac{\kappa^2}{(1+\kappa)^2}} - \mu_0\right)^{-1} \end{aligned}$$

Note that the above expressions define an equilibrium of the game if and only if they are feasible, i.e., $\hat{\mu} \in (\frac{\kappa}{1+\kappa}, 1]$. This is the case when $\mu_0 \in [0, \frac{1}{2}(1 - \frac{\kappa^2}{(1+\kappa)^2})]$.

Secondly assume that $p_1 > 0$. Then the equilibrium strategies solve

$$\begin{cases} \mu_0 = (1-p_0)\frac{\frac{\kappa}{1+\kappa}+\hat{\mu}}{2} \\ \hat{\mu} = \frac{\kappa}{(1+\kappa)p_0} \\ 1 - \frac{p_1}{2} = \frac{\kappa+1}{\kappa}p_0 p_0 + G(\hat{\mu}) + p_1 = 1 \end{cases}$$

This is a system of four equations in four unknowns. It has a unique solution (since $p_0 \in (0,1)$). Hence

$$\begin{aligned} p_0 &= \frac{-\kappa-(1-\mu_0)(\kappa+1)}{\kappa+2} + \frac{\kappa+1}{\kappa+2}\sqrt{\left(\frac{\kappa}{1+\kappa}\right)^2 + 3\left(\frac{\kappa+2}{1+\kappa}\right)^2 + (1-\mu_0)^2 + 2(1-\mu_0)\frac{\kappa}{1+\kappa}} \\ p_1 &= 2\left(1 - \frac{\kappa}{1+\kappa}\right) \\ G(\hat{\mu}) &= 1 - p_0 - p_1 \\ \hat{\mu} &= \frac{\kappa}{(1+\kappa)p_0} \end{aligned}$$

Note that the above expressions define an equilibrium of the game if and only if they are feasible, i.e., $\hat{\mu} \in (\frac{\kappa}{1+\kappa}, 1)$. This is the case when $\mu_0 \in (\frac{1}{2}(1 - \frac{\kappa^2}{(1+\kappa)^2}), \frac{2}{\kappa+2})$.

Finally suppose that $p_0 + p_1 = 1$, i.e., firms play full revelation. Note that p_1 and p_0 are directly characterized by the Bayes-plausibility constraint.

$$\begin{cases} p_1 = \mu_0 \\ p_0 = 1 - \mu_0 \end{cases}$$

This is an equilibrium if no deviation is profitable. Obviously the best deviation a firm can play is moving all the mass at 1 to $\frac{\kappa}{1+\kappa}$ and increasing the probability to preserve Bayes-plausibility. Hence it is an equilibrium if and only if

$$p\left(1 - \mu_0 + \frac{\mu_0}{2}\right) \geq p\mu_0 \frac{\kappa}{1+\kappa}(1 - \mu_0)$$
$$\Leftrightarrow \mu_0 \geq \frac{2}{1+\kappa}$$

Then full revelation is an equilibrium of the game if and only if $\mu_0 \in [\frac{2}{1+\kappa}, 1]$.

The three cases detailed above fully characterize the equilibrium of the information game between the two wood producers. $\square$