Savings in a 3-Period Model with a Behavioral Agent

Rational inattention with a Sparse Dynamic Approach

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Abstract

I study a sparse rational inattention 3 period model with a partial attention future income shock, I propose to introduce a partial attention interest rate to the same period of the income shock. When partial attention is considered the sparse agent divides itself in two, one is fully rational with the initial endowment, it shows patience and is able to smooth consumption, the other half of the agent has partial attention, due to her short sight both her estimates of interest rate and future income mare incorrect. To navigate the world the agent derives an euler equation, however this is based on perception, thus her euler fails to predict the behavior of this agent but she stills exhaust the budget with the actual consumption, given instead by the dynamic policy rule.

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Introduction

More often than not we think of the economic agent as capable of managing, at least up to some point, his resources in an efficient way, and actually obtaining the greatest benefit with the least expense. This idea is very well established in the economic science up to the point in which opening any economics book will reveal some sort of maximization or optimization behavior.

It is true that optimization is very useful tool in order to model behavior, without maximization we as scientist studying the human behavior would be lost without knowing what choice is at least, the most likely to be taken. In an essay by John Stuart Mill (1844, 144) refers to this very phenomenon:

> [...] Political Economy presuppose an arbitrary definition of man, as a being who invariably does that by which he may obtain the greatest amount of necessaries, conveniences, and luxuries, with the smallest quantity of labour and physical self-denial [...]

Notice that this interpretation identifies the the agent as a an optimizer; this individual could be easily described by a utility maximization problem.

But more importantly, it brings the intuition of a trade-off between benefit and cost; the trade-off is central to understand choice, especially if it is optimal, since it means that the choice maker has been able to select the actions that will lead to his best desirable outcome while keeping his discomfort at minimum. This task of choosing optimally is is not trivial, otherwise we would not even bother to think about choice. Nevertheless the optimal choice problem, actually, seems to be the opposite, requiring enormous quantities of cognitive power that seems unrealistic that an everyday person would even be willing, or much less, capable of devoting that kind of resources in the decision making process.

Maybe, as Simon (1955) thought, the agent in reality is not even aware of the fact that an optimization is being taken place in order to reach a decision, or maybe the brain is more skilled than we know to make choices. Whatever the case may be, it is at least worth consider the idea that real agents want to maximize outcome, but are not burdened by thousands of variables but instead make choices using easier rules of thumb or cognitive shortcuts, what we called heuristics.

This idea inspires the concept of behavioral economics, on one side the real world is a complex system of interactions and individuals, but on the other side we navigate through it without making specialized computations every single move. In particular the idea of having in reality thousands of variables but neglecting most of them in favor of just a few of them that are more important is what is called sparsity.

In the present document I study the recent development in the field of rational inattention using sparsity as a modeling tool made by professor Xavier Gabaix. The approach essentially consists in optimally assigning attention to those parameters that are more untactful in the objective function. In order to do that the method defines a 2 stages optimization where the first stage is to chose attention and the second to solve the base maximization program.

After studying the method, I propose to introduce a partial attention interest rate into a 3 period model developed by the author. Firstly we modeled interest rate as an anchoring process, parting from a default value R_d. This interest rate will interact no only with the variables of the model but also with a partial attention future income shock.

I compare the results with the base 3-period model. The sparse agent can be seen a 2 separate individuals sharing the same intertemporal budget constraint; on the one hand, the sparse agent is fully rational with the initial endowment, it shows patience and is able to perfectly smooth consumption over the three periods, on the other hand the it is partially myopic to the future income shock and also discounts with the perceived interest rate instead of the actual one.

Through sparse behavioral inattention we could derive an euler equation for an intertemporal problem, however this is nos longer a good way to characterize the agents behavior, because the actual consumption depends on the realization of the world in the next period, up until then the agent is partial myopic. in Since the income shock is in the future, the interest rate decreases the price of inattention in the income variable.

In regard to the attention function, the attention will depend on the magnitude of the parameter and the curvature of the objective function, I argue that the shape of this function is intuitive but not without its issues. Also there is no interaction term between attention variables, so the trade-off of assigning attention to certain variable, does not affect others.

This document is organized in the following way, the first section will present some basic concepts on behavioral economics, will elaborate on a discussion about rationality and bounded rationality and will provide some key concepts to understand the sparse behavioral inattention model. On section 2 we revisit the method proposed by professor Gabaix and try to structure all the basic insights necessary to develop the model. On section 3 we develop the model, first I present the 3 period model as a reference and later introduce the interest rate, the last part of this section will conclude on the model solved, section 4 will add some conclusions and final thoughts.

1 Background concepts

1.1 Rationality

A rational individual is not difficult to imagine in modern day economics, a typical economical agent will maximize his utility over some monetary constraint, and this idea is profoundly settled into the core of economics. Economist Gary Becker (1976) thought that one of the key elements of rationality was the maximizing behavior, this intuition was shared of course by a wide range of economist and other scientist as Becker himself pointed out. "It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest" says Adam

(?, 20) clearly implying a maximizing behavior of the butcher, the brewer or the baker.

The type of behavior we expect from a *homo economicus* where $F(\cdot)$ could be a production function or a price system transformation is the following:

$$\max_{Z} \quad U\left(z_1 ... z_L\right) \quad \text{subject to} \quad F\left(z_1 ... z_L\right) \leq T \tag{1}$$

This maximizing behavior is related to scarcity concludes Becker, this or any type of behavior, but importantly the existence of markets allowed for the scarce resources to be efficiently allocated, but also for the scarcity to be perceived by the agents in society. Compare this insight to a more modern approach to rationality by Amartya Sen's (2017) which refers to rational behavior to a maximizing conduct over a set of preferences and requiring internal consistency, this means that this rational behavior must correspond to others' rational behavior over their corresponding sets of preferences.

Contrasting both concepts we can see that both require maximizing behavior to exist beyond an individual level into a societal scale. This is interesting because the full intuition on rationality by Becker (1976) states three elements, (1) maximizing behavior over a set of (2) stable preferences and the (3) existence or at least assumption of markets. Markets will ensure internal consistency from the individual to the aggregate level and stable preferences allow the markets to remain efficient.

For our purposes we will refer as Rational Behavior to a given behavior which is the result of an optimization over a set of preferences, we will disregard the effect of the market since our analysis is on partial equilibria. In a program like (1) we will refer to the main unit of decision as rational then.

On the other hand, rationality will refer also to an optimizing behavior which takes into account all available information over the set of preferences (Becker, 1976). Notice a depart from the traditional microeconomic concept of rationality given by completeness and transitivity properties over the set of preferences; we will implicitly assume this kind of rationality for all of our models, although they could be build in more general setups.

Is important to note that an agent might not be even aware of the fact that a maximization is being conducted as pointed out by Friedman (1953); Becker (1976). Such unawareness will later be consistent with biases of thought of which a behavioral version of a rational economic agent is prone to.[1]

[1]Becker (1976) indicates that the optimizing behavior in the economic approach need not to be on a complete information or costless transactions setup, however, some scientist

Finally assume the utility and the transformation functions are increasing in every z_i and T is non negative in (1). Then, there is a clear trade-off between the maximization objective and the constraint, an increase in any z_i implies less available resources to transform $(T - z_i)$. This trade-off is important as a tool because the rational agent manages to minimize the loss while obtaining the maximum benefit.

1.2 Bounded Rationality

While most of the economics framework has been developed under the assumption of rationality. Some theories challenge rationality as a viable or even believable assumption at all. Hence, bounded rationality could be seen as an alternative way of thinking choice in economic agents. This concept was first proposed by Simon (1955), We remark that in this framework that behavioral agents are not irrational (choosing a non optimal outcome)[2], but instead should be thought as rational entities of choice over a (simplified) set of alternatives bounded by the their computational limitations.

In simple words, human beings do not make all the necessary computations, like assigning pay-offs and computing probabilistic outcomes before every single decision; as a matter of fact, Simon claims that no computation might have taken place at all in the first place. Instead, continues Simon, The agent may rely on computational simplifications that produce an approximate model for the choice process under evaluation (Simon, 1955). [3] This type of model could defy how economics draw conclusions and in turn how policies are made. Something similar to Keynes' contribution to economics as it is recalled by Cristopher Sims (2003, 665):

> Keynes's seminal idea was to trace out the equilibrium implications of the hypothesis that markets did not function the way a seamless model of continuously optimizing agents, interacting in continuously clearing markets would suggest. His formal device, price "stickiness" is still controversial, but those critics of it who fault it for being inconsistent with the assumption of continuously optimizing agents interacting in continuously clearing

use these market flaws to draw conclusion on irrational behavior which is a mistake.

[2]In modern behavioral economics there exist cases in which agents are irrational, however we are more interested in the former in the present study.

[3]This is an heuristic as we will define below.

markets miss the point. This is its appeal, not its weakness.

If true, the concept that agents are bounded but rational would, as price stickiness, imply that market is not the result of perfectly synced utility maximizing individuals, hence, possibly changing how conclusions are drawn and how policy is made. Meaning that models of all sorts (e.g. Dynamic optimization models) should be revisited with a new paradigm in hand. Sims (2003) proposes the inline concept of Rational Inattention, even though his approach focuses on an specific method, conceptually he formalizes the idea of bounded rationality as an optimization within an optimization. The central idea of Rational Inattention is the fact that agents have a limited information throughput (due to mental or informational limitations), in consequence, they optimize the use of limited bandwidth. Thus, only the most relevant information will be processed, later the agent will maximize the final outcome given the informational constraint.

The idea is to measure the transmission of information under uncertainty. The measurement of information transmitted will be the change in entropy, a stochastic variable X has a measure of entropy of: $-\mathbb{E}\left[log(p(X)\right]$ where $p(X)$ is the PDF of X. The maximum possible change in entropy given the shape of random variable will determine the bandwidth or the channel's capacity. Now say that X is a signal and Y is another random variable that represents our optimal choice (similar to a mixed strategy in a game), then we inform Y from the signal and then draw Y with some *pdf*; by building the difference in entropy between Y and the signal, in order to maximize the information received, that is the the resulting *strategy* will be the conditional distribution of Y given X.

1.3 Heuristics and Biases

One of the most notably advances in Behavioral Economics is Prospect Theory by Kahneman and Tversky (1979). These authors show that choices are not always optimal when risk, and therefore uncertainty, is involved. This non-optimality according to the authors, is due to the fact that people perceive more a loss than a gain, this is clearly a bias towards uncertainty in losses. [4]

[4] Prospect theory could be seen as an example of irrational behavior, since people is choosing a non-optimal outcome rather than choosing the best outcome within their bounded set of alternatives. However, it could be argued that there is a preference for risk

Recall from bounded rationality that people make choices based on simplifications of reality; We will define this simplifications as heuristics. This elements could take the form of rules of thumb or shortcuts in the model that draw conclusions more easily than making all computations (Kahneman, 2003). In its text about taking heuristics seriously, Gigerenzer (2016, 9) explain that "In AI, heuristics are used to make computers smart, yet in some corners of behavioral economics, heuristics are still seen as the reason why people aren't smart.". But in reality, heuristics could derive in errors as having optimal results, which maybe is a consequence of the unawareness of the agent that an optimization process is taking place as Friedman (1953) said, even in an heuristic process. This reinforces the idea that maybe "the unconscious is a better decision-maker than the conscious" (Simon, 1955, 104) in which case, heuristics offer the additional feature of requiring much less computational resources.

Now, heuristics could lead indeed to failure or bad predictions, and specially they make us susceptible to biases in the choice process (Samson, 2016). For our purposes we will define a (cognitive) bias in this context as a systematic error in the way a person thinks or analyses a certain choice problem (Ariely, 2008). This (systematic) error could be seen as a deviation from what could be considered a desirable outcome (e.g. rational choice).

One of the most important heuristic tools that people use to make decisions is anchoring. According to Tversky and Kahneman (1974) this is a process in which an individual would take a value as a reference point (an anchor), and then take a decision by adjusting from that value upwards or downwards accordingly. This reference value could be anything, like the previous know value of the interest rate, the average house price in the market or a given parameter as in the experiments by Tversky and Kahneman. Since anchoring leads to serious bias in prediction, it is a useful tool in modeling bounded rationality. Recall the maximization problem in (1), say for simplicity the shapes of the objective and constraint give us a convex maximization problem, furthermore assume that:

$$\exists \, \phi \, (z_1 ... z_L) \quad s.t. \quad \phi \, (z_1 ... z_L) = Z^*$$

This function $\phi(z_1 ... z_L)$ is the optimal action of (1). So, a behavioral action $\hat{\phi}(\cdot)$ could be thought as a deviation from some anchor value. This default value could be anything, like the action taken in the previous period

(risk aversion), in which case this agents are perfectly rational.

or a suggestion made by a social planner. Then, the behavioral action will be a convex combination of the optimal action and the default action:

$$\hat{\phi(\cdot)} = m\phi\,(z_1...z_L) + (1-m)\phi^d \tag{2}$$

Where the dampening factor $m \in [0,1]$ is the modeling tool that captures the adjustment from the default value of the action towards the optimal action. $m = 1$ will imply a perfectly rational agent. This formulation, with some variants or generalizations, is the main flavor to model bounded rationality as Gabaix (2017a) indicates. What is interesting of this formulation is that, unless $m = 1$, the adjustment done by the agents will never be sufficient, this is consistent with Tversky and Kahneman (1974) about insufficient adjustment being typical.

2 Sparsity as rational inattention

2.1 Principles

Entropy based informational economics have very nice properties in general but it could be difficult to deal with the algebra and build a tractable model. Within the contexts of bounded rationality, that is agents that build simplifications of the world around them and use heuristics to navigate it, and rational inattention we would like to model behavior in a way that resembles existing microeconomics theory. From the latter (context), we expect to see agents conducting two stages of behavior optimization, one to maximize their limited attention, and one to maximize outcome given limited attention. Xavier Gabaix proposes the use of the sparsity[5] notion to achieve such goal; an sparse individual will have many parameters to consider but will be attentive to only a handful of them (Gabaix, 2014a, 1662):

> For any decision, in principle, thousands of considerations are relevant to the agent: his income, but also GDP growth in his country, the interest rate, recent progress in the construction of plastics, interest rates in Hungary, the state of the Amazonian

[5]Sparse could mean diverse or scattered which is not much indicative of this agent's behavior; thus, it is more useful to think of sparsity in computational terms: a sparse array has very few non-zero elements. Then, all zeroes are factors to which the agent is inattentive.

> forest, and so on. Because it would be too burdensome to take all of these variables into account, he is going to discard most of them.

It is important to note that in principle, we could assigned a fixed *cognitive* cost to each of these parameters and given an attention upper-bound let the agent run a cost minimization, similar to a minimizing expenditure scenario where prices are cognitive costs and wealth or initial endowment is the attention upper-bound. Nevertheless, "fixed costs, with their nonconvexity, are notoriously ill-behaved" (Gabaix, 2014a, 1663). Thus, a modeling tool that do not lead to discontinuities and non-convexity issues which are difficult to deal with is much preferred. On the contrary, sparsity "entails well-behaved, convex maximization problems" (Gabaix, 2014a, 1662).

To start modeling the sparse agent behavior lets focus first on the final stage of optimization. Following Gabaix (2014a)[6], let us consider a traditional utility maximization problem as follows:

$$\max_{a} \quad U\left(a, x\right) \quad \text{subject to} \quad b\left(a, x\right) \geq 0 \tag{3}$$

Both a and x are vectors of arbitrary length, and $b(\cdot)$ is some budget constraint. a is the optimizing action that our agent must take to maximize her outcome, and x is the vector of parameters to which the agent can be attentive to, we will refer to them as parameters of attention. Finally, m is a vector of same length as x, every m_i is between 0 and 1 and these are the choice variables of attention, they could be called variables or levels of attention, or just attention. Now, lets assume that $\exists\, m^*$ and is the solution to the first stage optimization (*i.e.* is the optimal attention vector), then the problem will transforms as follows:

$$\max_{a} \quad U\left(a, x^s\right) \quad \text{subject to} \quad b\left(a, x^s\right) \geq 0 \tag{4}$$

[6]This author is the creator and, at the best of my knowledge, the only source of documentation on the model. For this reason, during the rest of this section I will elaborate on the author's work and draw heavily from his literature(*mainly* Gabaix, 2014a, 2017a,b). Some references may be omitted for the sake of reading and writing, but do note that most of the ideas and the development of the model in this section are not my doing

2.2 The attention function

Where $x^s := m^* x$. Notice that this is a known optimization problem, thus we know how to solve it already using microeconomics and optimization theory and the real problem in hand is how to chose m^*. There is only one difference in this stage from a traditional maximization problem, and that is the budget is taking *perceived* instead of actual parameters, but in order to understand how to choose m is better if we deal with such issue later so for a moment we will assume an unconstrained problem.

Since the agent is inattentive her outcome will deviate from the fully rational one. Call $v(m)$ to the utility that the agent obtains given m level of attention, ι will be the reference level of attention like the perfectly attentive case or some default. Then $v(m) - v(\iota)$ will be the differential of utility due to the deviation from the reference case. Then by *Lemma 2* (Gabaix, 2014a, 1700) the utility losses from attention deviation from reference can be approximated by Taylor (to a second degree)[7] with the following shape:

$$\mathbb{E}\left[v(m) - v(\iota)\right] = -\frac{1}{2}(m-\iota)^t \Lambda (m-\iota) + o\left(\|x^2\|\right)$$

The cost of inattention $\Lambda_{i,j} = -\sigma_{i,j} a_{x_i} u_{aa} a_{x_j}$ with $\sigma_{i,j} = \mathbb{E}\left[x_i x_j\right]$ is a very intuitive expression. The first component σ measures the correlation between parameters, a_x is the marginal impact in the optimal action of a change in the attention parameter x and u_{aa} measures the curvature of the utility function, hence, how much utility will be loss (or gained) by a change in the optimal action a induced by a change in attention. Since sparse agents rely on anchoring, then all derivatives are taken around this default value (typically fully inattentive), also the function above is a Taylor approximation, so is important that the derivatives are taken in a vicinity of $x = 0$ or the corresponding reference value.

It is important to note that the curvature of the objective and the marginal impact of the parameters on the optimal action are intimately linked: it is defined by:

$$a_{x_i} = \frac{\partial a}{\partial x_i} = -u_{aa}^{-1} u_{a,x_i}$$

The last term of this identity is the curvature of the objective function both in the action and the parameter. If we compare (1) and (3) we could

[7] for a proof of the lemma and further insight on the derivation of the cost of inattention check (Gabaix, 2014b)

see that there is not much difference between the program of a rational agent and of an sparse one; in particular both share an objective function u that takes both actions and parameters as input, hence we could have a function $v(a, x, m)$ that takes additionally attention as an argument, this shadow function v mimics up to a degree the real (rational) objective function into the rational inattention world, so the behavioral agent maximizes over his perception v. We would expect that:

$$v\left(a, x, m\right) = u\left(a, m \cdot x\right) \tag{5}$$

Where $(\cdot)$ is a component by component vector product[8]. If (5)is true, then Gabaix (2017b) proposes an additional characterization of the marginal impact of parameters on actions:

$$a_{m_i} = \frac{\partial a}{\partial m_i} = a_{x_i}\Big|_{m=(1\ldots1),x=0} x_i \tag{6}$$

Notice that the expression for the utility differentials $\left(\mathbb{E}\left[v(m) - v(\iota)\right]\right)$ resembles a quadratic loss function, so if we assume $o\left(\|x^2\|\right)$ is negligible and add a cost κ we obtain an objective function from which m^* will minimize both cognitive cost and utility loss, hence:

$$\begin{aligned} m_i^* \quad &= \quad \underset{m\in[0,1]}{\operatorname{argmin}} \quad \frac{1}{2}\sum_{i,j}(1-m_i)\Lambda_{i,j}(1-m_j) + \kappa\sum_i m_i^\alpha \\ &\qquad\qquad \text{where} \quad \Lambda_{i,j} = -\sigma_{i,j}a_{x_i}u_{aa}a_{x_j} \end{aligned} \tag{7}$$

Lets define a function: $m^* = \mathcal{A}\left(\frac{\sigma^2}{\kappa}\right)$, the quotient argument captures both variance and cost which will affect the shape of the attention function and hence the resulting attention level. The cognition cost measures the effort of taking into account much information while making a decision and it is "a taste for sparsity. When $\kappa = 0$ the agent is the traditional agent" (Gabaix, 2017b, 10). In the *Lemma 1* (Gabaix, 2014a, 1672) the author states that only when the cognitive cost is linear in the attention parameters, then the attention function $\mathcal{A}$ induces "both sparsity and continuity". Which is the reason why Gabaix suggest using linear costs in most application. Finally the author proposes an easy formula for the attention function with linear costs:

[8]Later on this document we will use indistinctively the notation mx, that given the appropiate context, will represent the component by component vector product

$$\mathcal{A}\left(\frac{\sigma_i^2}{\kappa}\right) = max\left(1 - \frac{1}{\sigma_i^2}, 0\right) \tag{8}$$

If (5) holds true, then: $\Lambda_{i,j} = -\mathbb{E}[a_{m_i} u_{aa} a_{m_j}]$. All derivatives are evaluated at m^d typically $m^d = (1...1)$ and a^d. In this scenario $m_i^* = \mathcal{A}(\mathbb{E}[a_{m_i}u_{aa}a_{m_j}]/\kappa)$. Equivalently using the other set of derivatives: $m_i^* = \mathcal{A}(-\sigma_i^2 a_{x_i} u_{aa} a_{x_j}/\kappa)$.

2.3 the sparse operator

As given already by our definition of bounded rationality, the sparse agent will have two stages of maximization. The above is the first stage consisting in choosing attention optimally, and the second stage will be solving for the optimal action with the chosen attention. This algorithm is denoted by Gabaix (2014a) as *Definition 1* and it will be the main algorithm for solving sparse rational inattention behavior, thus formally:

Sparse-max operator unconstrained: Define the *smax* operator as the following algorithm in order to solve the program: $\text{smax}_{a,m}\, v\,(a, x)$

Step 1: $m_i^* = \mathcal{A}(\sigma_i^2/\kappa) = \underset{m\in[0,1]}{\text{argmin}} \quad \frac{1}{2}\sum_{i,j}(1-m_i)\Lambda_{i,j}(1-m_j) + \kappa\sum_i m_i^\alpha$

Step 2: *Solve* $a* = \arg\max_a v(a, x^s)$. with $x_i^s = m_i^* x_i \;\; \forall i = 1...n$

Now, recall the main optimization problem (4) in which the agent takes the action that maximizes the outcome subject to a budgetary constraint. Say that x is a price or an income shock, therefore the perceived budget in which the agent chooses her optimal behavior is not consistent, and it may lead to non-exhaustion of the budget, or even worse, over-consumption. Th force the agent to take into account the budget constraint we transform the program into an unconstrained one using a Lagrangian $\mathcal{L}$ which is now the new objective function. By building a Lagrangian, the agent will tune the multiplier to ensure that the budget constraint binds (Gabaix, 2014a).

Sparse-max operator allowing for constraints: Define the *smax* operator as the following algorithm in order to solve the program:

$$\underset{a,m}{\text{smax}}\, v\,(a, x) \quad \text{subject to} \quad b(a, x) \geq 0$$

Step 1: Build the Lagrangian: $\mathcal{L}(a, x) = v\,(a, x) + \lambda b(a, x)$

Step 2: Solve the model using $\mathcal{L}(a,x)$ as new objective and following the *smax* unconstrained operator algorithm.

2.4 Dynamic programming

Lets start with a familiar setup, a rational agent will maximize:

$$\begin{aligned} \max_{a_t} \quad & \sum_{t_0}^{T-1} \beta^t \, u\left(a_t, z_t\right) \\ \text{s.t.} \quad & z_{t+1} = F\left(a_t, z_t, \epsilon_{t+1}\right) \end{aligned}$$

Where $F\left(\cdot\right)$ is a law of motion. Equivalently we can write this problem to a corresponding Bellman equation as follows:

$$V^{t,r}(z) \quad = \quad \max_{a} \; u\left(a,z\right) + \beta \, \mathbb{E}\left[V^{t+1,r}\left(F\left(a_t, z_t, \epsilon_{t+1}\right)\right)\right]$$

dynamic sparse-max operator: the optimal action given by the dynamic *smax* operator is such that:

$$\begin{aligned} a^s(z, V^{t,p}) \quad &= \quad \underset{a,m|m^d}{\arg\,\mathrm{smax}} u\left(a,z,m\right) + \beta \, \mathbb{E}\left[V^{t+1,p}\left(F\left(a_t, z_t, \epsilon_{t+1}, m\right)\right)\right] \\ & \qquad \text{where} \quad V^{t+1,p} \quad \text{is the proxy (anchor)} \end{aligned} \tag{9}$$

Similarly as in the case of static optimization with sparse agents, we define a utility function to mimic the actual utility function, $u(a,z,m)$; additionally we define a similar function for the law of motion $F\left(a_t, z_t, \epsilon_{t+1}, m\right)$. We also define a proxy value from which the agent's value function will deviate, typically this proxy value will be: $V^{t,p}(\cdot) = V^{t,r}(\cdot)$, this assumption will imply that the agent projects herself as rational in the future, or more realistically, she fails to see her inattention.

The concepts regarding the dynamic optimization will be clearer in the 3-period example in the following section. A key element of the dynamic approach is that the anchoring point is rationality, moreover we will learn that this result is central in resolving rational inattention model with sparse agents.

3 A behavioral agent in action

We will study a cake eating model with partial inattention in both an income shock and the interest rate. The model consist in 3 periods. The utility is $\sum_{t=0}^{2} u(c_t)$ with no discounting. There is an income shock of magnitude x_w in the last period and an interest rate R between periods 1 and 2 that will be perceived as a deviation from a default interest rate.

3.1 A 3-period example

Before solving the model described we revisit a simpler 3-periods model proposed by Gabaix (2017b) in which there is no interest rate with agent's full awareness of it (i.e $R^d = 1$). Thus, there is not additional incentive to save for retirement other than to attempt to smooth consumption and the budget constraints for periods 0, 1 and 2 respectively are: $c_0 + w_1 \leq w_0,\ c_1 + w_2 \leq w_1\ and\ c_2 \leq w_2 + x$. This primer example will allow us to understand the consumption profile of an sparse agent and provide some useful insight on how to solve the algebra. A rational agent in this scenario, since there is not discounting nor interest rate, will perfectly smooth consumption, so she will distribute her lifetime income $(w_0 + x)$ among all three periods:

$$c_t^r = \frac{w_0 + x}{3} \qquad \forall t = 0, 1, 2$$

As Gabaix (2017b) indicates, a dynamic policy version of this behavior is as follows:

$$c_0^r = \frac{w_0 + x}{3}, \qquad c_1^r = \frac{w_1 + x}{2}, \qquad c_2^r = w_2 + x$$

Notice that this policy is intuitive, as at time 1, for example, the agent equally divides the remaining income among the remaining two periods; in the same way during the final period ($t = 2$) the agent simply consumes all her available resources.

Since there is only one parameter to be attentive to, we will drop the subscript in x. Now, we introduce partial attention in income with the variable m_t, hence, the sparse agent perceives the income shock at time t by: $x_t^s = m_t x$. So a fully inattentive agent $m_w = 0$ will be completely unaware of the income shock, on the contrary a fully attentive agent is rational in the income shock variable.

Time 2: The agent maximizes only his present consumption and no attention is required since the income shock has been realized already:

$$V^2(w_2, x) = \max_{c_2} \quad u(c_2) \quad \text{subject to} \quad c_2 \leq w_2 + x$$

The agent will consume all her available income, hence:

$$c_2 = w_2 + x$$

Time 1: The maximization problem is:

$$\begin{array}{lll} \underset{c_1, w_2, m_1}{\text{smax}} & \hat{v^1}(c_1, w_2, w_1, x, m_1) = u(c_1) + V^2(w_2, x) & \\ \text{s.t.} & c_1 + w_2 \leq w_1 & \\ & m_1 = m_1^* & \textit{i.e.} \text{ optimal attention level} \end{array}$$

Following the *smax* operator definition we should choose attention first, however we will see that assuming m_1 optimal as given for the moment is more practical. Then, we transform the problem into an unconstrained optimization[9] with optimal attention, thus the problem is no longer an *smax*:

$$\max_{c_1} \quad v^1(c_1, w_1, x, m_1) = u(c_1) + V^2(w_1 - c_1, m_1 x)$$

By FOC $= 0$ we obtain: $u'(c_1) = V_w^2(w_1 - c_1, m_1 x) = u'(w_1 - c_1 + m_1 x)$, Thus:

$$c_1 = w_1 - c_1 + m_1 x \quad \Longrightarrow \quad c_1 = \frac{w_1 + m_1 x}{2} \tag{10}$$

Thus, the default consumption is $c_1^d = c_1|_{m_1=0} = \frac{w_1}{2}$. Now, with a functional form for the policy rule we can compute more easily the cost of inattention which is needed to choose m_1 optimally. Recall from (7) that the cost of inatention is given by the curvature of the objective function:

$$v_{cc}^1(\cdot)\Big|_{c_2^d} = u''(c_2^d) + V_{ww}^{2,p}(w_1 - c_1, m_1 x)\Big|_{c_2^d}$$

We assume, following (9), that the agent will project herself as rational in the next period, that is: $V^{2,p}(w_1 - c_1, m_1 x) = V^{2,r}(\cdot) = u(c_2^r)$. Notice

[9] we assume nice properties on the function u so the constraint is saturated, hence no need to build a Lagrangian

that $c_2^r = w_2 + x$, consequently $V_{ww}^{2,p} = u''(c_2^r)$. We have then, the curvature of the objective is:

$$v_{cc}^1(\cdot)\big|_{c_1^d} = 2u''(c_1^d) \tag{11}$$

Finally, to choose m_1 optimally we take linear cognitive cost κ as suggested by the author (Gabaix, 2014a, 2017b). Then, the attention function $\mathcal{A}(\cdot)$ can be characterized by the easy formula in (8) with the following arguments:

$$m_1 = \mathcal{A}\left(\frac{1}{\kappa} v_{cc}^1 \operatorname{var}\left(\frac{\partial c_1}{\partial m_1}\right)\right) = \mathcal{A}\left(\frac{1}{2\kappa} u''(c_1^d)\,\sigma_x^2\right) \tag{12}$$

Time 0: For the last period the agent's problem is:

$$\begin{aligned}
&\underset{c_0,w_1,m_0}{\text{smax}} && \hat{v}^0(c_0, w_1, w_0, x, m_0) = u(c_0) + V^1(w_1, x) \\
&\text{s.t.} && c_0 + w_1 \leq w_0 \\
& && m_0 = m_0^* \qquad i.e. \text{ optimal attention level}
\end{aligned}$$

Following the same procedure as before the problem above becomes the following unconstrained optimization:

$$\max_{c_0} \quad v^0(c_0, w_0, x, m_0) = u(c_0) + V^1(w_0 - c_0, m_0 x)$$

FOC $=0$ implies: $u'(c_0) = V_w^{1,p}(w_0 - c_0, m_0 x) = u'\left(\frac{w_0 - c_0 + m_0 x}{2}\right)$ using (10), Hence:

$$c_0 = \frac{w_0 + m_0 x}{3} \tag{13}$$

Following, the default consumption is $c_0^d = \frac{w_0}{3}$. To compute the curvature we use (9) again and obtain: $V^{1,p}(w_0 - c_0, m_0 x) = V^{1,r}(\cdot) = u(c_1^r) + u(c_2^r)$. In the rational case the consumption is perfectly smooth so: $V^{1,r}(\cdot) = 2u(c_1^r)$, therefore $V_{ww}^{1,p} = \frac{1}{2}u''\left(\frac{w_1+x}{2}\right)$. With all of the above we get the curvature of the objective:

$$v_{cc}^0(\cdot)\big|_{c_0^d} = u''(c_0^d) + \frac{1}{2}u''(c_0^d) = \frac{3}{2}u''(c_0^d) \tag{14}$$

Finally, the optimal attention m_0 is:

$$m_0 = \mathcal{A}\left(\frac{1}{6\kappa}\, u''\left(c_0^d\right)\sigma_x^2\right) \tag{15}$$

The consumption policy function for the behavioral agent in terms of attention is the following:

$$c_0^s = \frac{w_0 + m_0 x}{3}, \qquad c_1^s = \frac{w_0}{3} + \left(\frac{m_1}{2} - \frac{m_0}{6}\right) x, \qquad c_2^s = \frac{w_0}{3} + \left(1 - \frac{m_1 + m_0}{2}\right) x \tag{16}$$

By comparing (12) and (15) we have that

$$m_0 \leq m_1 \;\; iff \;\; \frac{1}{6}\left|u''\left(\frac{w_0}{3}\right)\right| \leq \frac{1}{2}\left|u''\left(\frac{w_1}{2}\right)\right|$$

This is true trivially in $x = 0$ because is the rational case and as long x is not to large since w_1 is increasing in x.

We can conclude that sparse agents are patient as rational agents, that is they do not spend everything in one period, however they are myopic to future changes. Given the shape of the attention function in particular, the smaller the change in the future, the more myopic the agents will be. It is interesting that on one side the agents have been able to fully smooth the wealth through all 3 periods like is evident in (16), and at the same time the are myopic to the income shock.

Notice an important feature about the sparse agents, namely the ability to fulfill the perceived euler equation but fail the actual euler. (Gabaix, 2017b) points out that euler equations are a poor way of modeling behavioral agents, whereas sparse consumption functions are much more robust, specially since they can adapt to any circumstance, including the rational and fully irrational cases.

3.2 Introducing interest rate to an sparse agent

After solving the previous example we can build on those results to insert an interest rate ($i.eR > 1$). We have two attention variables this time, so we retrieve the subscript in x_w to denote the income shock. On the other hand, the interest rate will use the concept of anchoring so the actual interest rate is $R = R^d + x_R$; since R will be applicable only to the second period savings, all the budget constraints remain the same but the one from period 2 that becomes: $c_2 \leq Rw_2 + x$. Similar to the previous example we should first visit the rational agent. We expect smooth consumption over periods 0 and 1 due to free transfer of resources and the agent will consume all her remaining resources in period 2. The dynamic policy correspondingly is:

$$c_0^r = \frac{w_0 + {}^{x_w}/_R}{3}, \qquad c_1^r = \frac{w_1 + {}^{x_w}/_R}{2}, \qquad c_2^r = Rw_2 + x$$

In contrast to the rational agent, the sparse agent will optimize a different program. On the income side, partial attention will be modeled the same way as in the previous example using now the variable $m_{w,t}$. To model partial attention to the interest rate the agent will start from a default value and adjust her perception given attention $m_{R,t}$; hence the perceived interest rate is: $R_t^s = R^d + m_{R,t} x_R$. To shorten notation we define the vectors $\bar{x} := (x_R, x_w)$ and $\bar{m}_t := (m_{R,t}, m_{w,t})$.

Time 2: Now the optimization problem of this agent will look like:

$$V^2(c_2, w_2, \bar{x}) = \max_{c_2} \quad u(c_2) \quad \text{subject to} \quad c_2 \leq Rw_2 + x_w.$$

We know already that the agent at time 2 will simply consume all her available income:

$$c_2 = Rw_2 + x_w = (R^d + x_R)w_2 + x_w \tag{17}$$

Time 1: Now we need to consider attention variables ($\bar{m}_t$) for both parameters ($\bar{x}$). The problem to solve is:

$$\begin{aligned} &\underset{c_1, w_2, \bar{m}_1}{\text{smax}} && \hat{v^1}(c_1, w_2, w_1, \bar{x}, \bar{m}_1) = u(c_1) + V^2(w_2, \bar{x}) \\ &\text{s.t.} && c_1 + w_2 \leq w_1 \\ & && \bar{m}_1 = \bar{m}_1{}^* \qquad \textit{i.e.} \text{ optimal attention level} \end{aligned}$$

As in the simpler example we assume $\bar{m}_1$ optimal and proceed with the optimization, hence, the unconstrained problem is:

$$\max_{c_1} \quad v^1(c_1, w_1, \bar{m}_1 \bar{x}) = u(c_1) + V^2(w_1 - c_1, m_{R,1} x_R, m_{w,1} x_w)$$

From FOC$[c_1] = 0$ we have an euler condition:

$$u'(c_1) = V_w^2(w_1 - c_1, m_{R,1} x_R, m_{w,1} x_w) = R_1^s \, u'(R_1^s (w_1 - c_1) + x_w^s)$$

This *perceived* euler equation cannot be solved analytically so we need to make a further assumption: $u(c_t) = \log(c_t) \quad \forall t = 0, 1, 2$. Hence, the FOC becomes:

$$\frac{1}{c_1} = \frac{R_1^s}{R_1^s(w_1 - c_1) + x_w^s} \implies c_1 = \frac{R_1^s w_1 + x_1^s}{2R_1^s} \tag{18}$$

Thus, the default consumption is $c_1^d = c_1|_{\bar{m}_1=0} = \frac{w_1}{2}$. Following the steps in the example we will assume that the agent will project herself as rational in the future, so we have: $V^{2,p}(w_1 - c_1, m_1 x) = V^{2,r}(\cdot)$, moreover:

$$V^{2,p}(\cdot) = u(c_2^r) = u'(R\,w_2 + x_w)\big|_{\bar{x}=\bar{m}_1\bar{x}}$$

To finish the first period we need to compute the attention variables' value, but here is where we depart from the previous example because the following identity is not valid anymore: $\frac{\partial c}{\partial x} = \frac{\partial c}{\partial m}x$, this is due to the non-linear interactions between the attention variables in the policy rule (18), this implies that the marginal impact on the action of the attention parameter x is not characterized by the partial derivative *w.r.t.* the attention level m; instead, the more general definition of the marginal impact c_x is given by the identity: $c_x^t = -(v_{cc}^t)^{-1}\, v_{cx}^t$.

Also the simplified formula provided in (Gabaix, 2017b) is no longer applicable and the general definition (7) needs to be applied. Firstly the curvature of the objective function is:

$$\begin{aligned} v_{cc}^1(\cdot)\big|_{c_1^d} &= u''(c_1^d) + V_{ww}^{2,p}(w_1 - c_1, m_1 x)\big|_{c_1^d} \\ &= u''(c_1^d) \quad R_1^s\, u''(R_1^s(w_2) + x_w^s)\big|_{c_1^d} \end{aligned}$$

$$v_{cc}^1(\cdot)\big|_{c_1^d} = (1 + R_1^s)\, u'' + \left(\frac{w1}{2}\right) \tag{19}$$

By the identity mentioned above, we need to compute the curvatures of the objective function on c and x. Thus, we get:

$$\begin{aligned} v_{c,x_R}^1(c_1, w_1, \bar{m}_1\bar{x}) &= R\,w_2 u''(c_2^r)\big|_{\bar{x}=\bar{m}_1\bar{x};c_1^d} \\ &= R^d\,(w_1 - c_1^d)u''(c_1^d) = \frac{R^d\,w_1}{2}u''\left(\frac{w1}{2}\right) \\ v_{c,x_w}^1(c_1, w_1, \bar{m}_1\bar{x}) &= R^2\,u''(c_2^r)\big|_{c_1^d} = R^{d^2}\,u''\left(\frac{w1}{2}\right) \end{aligned}$$

Costs of inattention are computed combining (19)with v^1_{c,x_w} and v^1_{c,x_R} from above such that: $\Lambda^t_x = -\sigma_x c^t_x u_{cc} a^t_x$:

$$m_{R,1} = \mathcal{A}\left(-\frac{\sigma^2_{x_R}}{\kappa}\,\frac{R^{d^2}\,w_1^2}{4(1+R^{d^2})}\,u''\left(\frac{w1}{2}\right)\right) \tag{21a}$$

$$m_{w,1} = \mathcal{A}\left(-\frac{\sigma^2_{x_w}}{\kappa}\,\frac{R^{d^2}}{1+R^{d^2}}\,u''\left(\frac{w1}{2}\right)\right) \tag{21b}$$

Time 0: For the last period, assuming $\bar{m_0}$ optimal, the unconstrained agent's problem is:

$$\max_{c_0} \quad v^1\left(c_0, w_0, \bar{m_0}\bar{x}\right) = u\left(c_0\right) + V^1\left(w_0 - c_0, m_{R,0}x_R, m_{w,0}x_w\right)$$

From FOC$[c_0] = 0$ we obtain the following euler: $u'\left(c_0\right) = 2\,u'\left(\frac{R^s_0(w_0-c_0)+x^s_{w,0}}{2R^s_0}\right)$. Using functional form of utility we have the following policy rule:

$$\frac{1}{2c_0} = \frac{R^s_0}{R^s_0(w_0-c_0)+x^s_{w,0}} \implies c_0 = \frac{w_0 + {}^{x^s_{w,0}}/_{R^s_0}}{3} \tag{22}$$

Thus, the default consumption is $c^d_0 = c_0|_{\bar{m_0}=0} = \frac{w_0}{3}$. Again the agent assumes herself rational in the future, thus:

$$\begin{aligned} V^{1,p}\left(\cdot\right) \;=\; V^{1,r}\left(\cdot\right) &= u\left(c^r_1\right) + u\left(c^r_2\right) \\ &= u\left(\frac{w_1 + {}^{x_w}/_R}{2}\right) + u\left(R\,w_2 + x_w\right) \end{aligned}$$

To finish the last period we need compute the arguments necessary to characterize the attention level program. The curvature of the objective function is:

$$\begin{aligned} v^0_{cc}\left(\cdot\right)\big|_{c^d_0} &= u''\left(c^d_0\right) + \frac{1}{4}u''\left(c^d_0\right) + R^{d^2}\,u''\left(c^d_0\right) \\ &= \left(\frac{5}{4} + R^{d^2}\right)u''\left(\frac{w_0}{3}\right) \end{aligned} \tag{23}$$

The curvatures of the objective function on c and x are:

$$v^0_{c,x_R}(c_0, w_0, \bar{m_0}\bar{x}) = \frac{R^d\,w_0}{3}u''\left(\frac{w0}{3}\right)$$

$$v^0_{c,x_w}(c_0, w_0, \bar{m_0}\bar{x}) \quad = \quad \left(\frac{1}{4R^d} + R^d\right) u'' \left(\frac{w0}{3}\right)$$

Costs of inattention are computed combining the curvature of the objective (23) and the latter results as follows: $\Lambda^t_x = -\sigma_x c^t_x u_{cc} c^t_x$ and using the characterization given by (8) we obtan:

$$m_{R,0} \;=\; \mathcal{A}\left(-\frac{\sigma^2_{x_R}}{\kappa}\left(\frac{{R^d}^2\, w_0^2}{9(5/4 + {R^d}^2)}\right) u''\left(\frac{w0}{3}\right)\right) \qquad , \qquad (25a)$$

$$m_{w,0} \;=\; \mathcal{A}\left(-\frac{\sigma^2_{x_w}}{\kappa}\left(\frac{1}{4R^d} + R^d\right)^2 \left(5/4 + {R^d}^2\right)^{-1} u''\left(\frac{w0}{3}\right)\right) \qquad (25b)$$

The consumption policy function for the behavioral agent in terms of attention is:

$$c^s_0 = \frac{w_0}{3} + \frac{x^s_{w,0}}{3R^s_0}, \qquad c^s_1 = \frac{w_0}{3} - \frac{x^s_{w,0}}{6R^s_0} + \frac{x^s_{w,1}}{2R^s_1}, \qquad c^s_2 = \left(\frac{w_0}{3} - \frac{x^s_{w,0}}{6R^s_0} - \frac{x^s_{w,1}}{2R^s_1}\right) R + x_w \qquad (26)$$

Consider the policy function above, the agent is able to behave rationally with respect to the investment, that is she smooths consumption during the first two periods and the third period it consumes savings plus interest. On the contrary the agent behaves myopic with respect to the income shock and the interest rate, this is intuitive since the agent deals with investment from the first period instead of having to anticipate it, so this sparse agent is fully rational and attentive to the elements at time 0 (or the period in which she lives). Adopting the functional logarithmic form of the utility we can see that the myopia increases with time, if the period is further in time, then the agent will be more myopic to the changes in it.

We cannot analyze income and interest rate separately because all wealth flows (especially the income shock) are discounted by the interest rate, this is interesting because an interest rate greater than 1 will actually decrease the price of inattention on the income shock due to discounting. Now, assume that both x_R and x_w are positive, now, lets follow the optimal choice of our agent. In the first period the agent, as we mentioned before behaves rationally with the information on her current period, thus she smooths consumption over the positive income shock in the future, however she perceives both the magnitude of the shock and the discount rate smaller than what they are. Each period she subtracts the share of wealth consumed in the period before, however the perceived income shock keeps growing so the most likely scenario is that this share of wealth consume with respect to the actual wealth (or the new perceived) keeps shrinking. Both c_1 and c_2 are decreasing in R, so a higher x_R or more attention to the interest rate are incentives to save more in the second period.

To analyze the attention we need to recall that the attention function is increasing in the argument, also that the argument is strictly positive. Notice that $sigma^2$ represents the magnitude of the attention parameter, so the attention is increasing in the magnitude of the x. Another important remark is that the attention in x_R is increasing in the amount of available wealth at the period, it comes from the budget constraint at period 2, so that means that more available resources increases the amount of possible savings.

Notice that all the attention functions have the same second derivative of the utility function evaluated at different default parameters, if we would take the rational case as the default notice that both c_0^d and c_1^d will increase in size because the income shock is now visible for the agents, this would imply that the derivative would be closer to 0, that is the derivative shrinks in absolute value, and so the argument would shrink. This means that an agent whose default is closer to the rational will have smaller attention levels according to this model.

4 Conclusions

Conclusions

The sparse behavioral agents are an interesting tool to model less than rational behavior. The simplicity of the model make it ideal to recreate most of the already existing results in economics as the author has done with several models, especially in microeconomics; and revisit already existing results from a behavioral perspective. The tractability is one of its main advantages and also the fact that is already a readily framework, with its *smax* operator, to model various kinds of choice theory scenarios. Especially the former means that this tool could be heavily draw from a computational point of view if the necessary results for convergence exist or could be derived. In fact the author does mention some contraction mapping theory which is a good sign.

The behavior modeled in this paper with sparse inattention produced some interesting and consistent results. The agents behave as rational actors during the current period but failed to anticipate future shocks seems to agree with the theory of tunneling or similar by which us, agents, would assign much more of our attention to the current period. Another consistent result is the fact that agents would save in case of negative income shock, but due to short-sightedness this savings will more likely be insufficient. The introduction of the interest rate also produce some satisfactory and consistent results like increasing the incentive to save for the agents and decreasing the price of attention to the income shock. A possible extension could be to study solely the effect of the interest rate or to further study dominance of income over interest rate.

The shape of the attention function is intuitive and is very relatable which is always desirable for a behavioral model. however, the Taylor approximation taken to compute a functional form for the attention works only in the vicinity of a small x, and we are always evaluating at the default, it may be incorrect to draw conclusions so far off the point where derivatives were taken.

Notice that the attention function takes all attention parameters at once and assign optimally attention to each of them in an individual manner. This may be convenient for computational purposes, but also implies that there is no interaction between the parameters and variables, especially there is no trade-off between assigning too much attention to one parameter versus the other. A solution to this will be to formulate a generalization of the attention function, taking into consideration the actual program to choose attention optimally; in this program we could have correlated variables and the objective function will become an interaction term between them. Nevertheless this is still a geometric interaction which not necessarily captures the idea of a trade-off.

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