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Whether and Where to Apply*

Information and Discrimination in Matching with Priority Scores

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Abstract

This paper considers a matching market where agents have private information on their priority scores and must choose an object to which they apply. The analysis derives the Bayes-Nash equilibria, computes welfare ex ante and interim, and discusses implications for market design. Three main findings emerge. One, there is no symmetric equilibrium in pure strategies. Second, the symmetric equilibrium exhibits a block structure: agents sort into a finite number of classes of neighboring scores where they use the same application strategy. Third, the inefficiencies proceeding from the frictional market design prove interim asymmetric: low-score agents are better off under private information than under public information. In total, private information mitigates the discriminatory power of the priority system.

Keywords: Matching markets, priorities, private information, discrimination.
JEL codes: D82, C78.

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1 Introduction

Priority scores

Matching problems study the formation of productive partnerships, with numerous applications to marriage, labor, housing, college admissions, organ donation, and many more. Important primitives of any matching problem are the preferences, which specify how market participants value each other or (non-strategic) objects, and priorities, which specify how participants value the objects.

Preferences and priorities can be expressed either cardinally or ordinally. Subsequently, allocation mechanisms can be ordinal (demanding that participants submit rank-ordered lists of the available options), or cardinal (demanding that participants reveal how much they like each available option). There is evidence that the mechanisms used on real-life markets are increasingly cardinal on the priority side. Market operators more and more use priority points and priority scores. Examples include college admissions, where students are assessed based on their scores in standardized tests, civil servant job markets such as teacher or doctor allocation, where civil servants accumulate points for experience or performance, or social housing, where households are assigned a score reflecting the emergency of the housing need.

This suggests that cardinality helps to finely discriminate between agents on the market.[1] We, micro-theorists must certainly accommodate the shift from ordinality to cardinality. We must model the effect of working with cardinal priorities, with two goals: to improve our understanding of real-life markets (this is the descriptive stake) and to engineer matching markets efficiently (this is the normative stake).

Private information

The issue is that the natural information structure on the priorities, absent any intervention by the market operator, is private information. An agent observes her own priority score but is ignorant of other agents' scores. This is because, in general, these scores are computed based on criteria of private circumstances. FIGURE 1 below illustrates this point.

City of London Allocations Scheme		Secondary Points																
		Overcrowding		Wellbeing						Unsuitable Housing Conditions						Housing Management		
Primary Group	Primary Points	Per room lacking	Mixed sharing	Medical			Welfare			Sharing			Lack of tenancy	Bedroom Cap	Long TA stay	Advice & Engagement	Intentionality	Decant Urgency
				S	M	L	S	M	L	F	1-4	5+						
Management Transfer	800																	
Under-occupation	400			50	25	10	50	25	10					50				100 / 200
Severe Medical / Welfare	275	25	10	50	25	10	50	25	10	5	10	15	5			15	minus 50	
Severe Overcrowding	250	25	10		25	10		25	10	5	10	15	5			15	minus 50	
Studio Upgrade	250	25			25	10		25	10									
Decants	225	25	10	50	25	10	50	25	10									100 / 200
Moderate Medical / Welfare	225	25	10		25	10		25	10	5	10	15	5				minus 50	
Moderate Overcrowding	200		10			10			10	5	10	15	5				minus 50	
Homeless	140	25	10	50	25	10	50	25	10						150		minus 50	
Lower Income City Connection	100					10			10	5	10	15	5					
Sons and Daughters	50					10			10									
Low Priority	1					10			10	5	10	15	5					

FIGURE 1: London social housing - Priority score computation rule

The table shows the scoring rule used in London social housing. Households are awarded priority points if they are currently homeless, have large families, or have health issues. Household A knows about her status with respect to every criterion and, when given this table, can compute her aggregate priority score. But household A could not make the same computation for another household, say B, simply because she has no idea about the inputs.

The empirical matching literature supports this intuition providing evidence that participants are poorly informed about their order in priority. Kapor, Neilson and Zimmerman (2020) [13] using data from New Haven, US, show that beliefs about admissions chances differ from rational expectations values. They predict choice behaviors and quantify the welfare costs of belief errors.

[1] Throughout the article, we use the word discrimination in a positive sense. Discriminating means allocating a good (with a high probability) to the people who need it the most.

Fabre et al. (2021) [9] using data from Chile show that on-the-fly information about programs' cutoff scores has a causal effect on reducing students' biases, application mistakes, and improving students' outcomes.

Theoretical matching papers most often do not investigate the effect of this private information.[2] Most papers either unrealistically assume perfect information or set strategy-proof mechanisms (hence, no incentive to know about others).
In practice, though, the most widely used mechanisms are typically truncations of standard strategy-proof mechanisms. Agents are not allowed to rank all available options and submit a truncated rank-ordered list. The issue is that the truncated versions of the mechanisms are no longer strategy-proof. The manipulation consists of listing safe objects.

Leading example: Social Housing in Europe

A prominent example is the assignment of social housing units in Paris. Since 2016, the municipality has allocated around 4,500 housing units a year through an online scheme called "*LOC'annonces*".[3] The allocation occurs in three steps. In the first step, households register as social housing seekers. The market operator performs eligibility checks and places households in rent and bedroom categories depending on their earnings and sizes. Most importantly, households are assigned priority scores based on their circumstances. The computation mode awards points for homelessness or unsuitable current housing (overcrowding), ill-health status, and more criteria. In the second step, households apply for vacant housing units. More specifically, vacancies are advertised on a dedicated website[4] from each Tuesday morning until the following Wednesday midnight. Households apply to one housing unit per round or choose not to apply. Very importantly, no precise feedback information is provided on the identities of other applicants to the same housing units. Following the application closing, applicants who have applied and who are on a shortlist of the highest priorities can view the accommodation and decide to maintain or withdraw their bids. In the third and final step, each vacant property goes to the applicant with the highest priority score among those who have applied for it. The whole allocation process, from the application closing to the final allocation, can take a maximum of three months.[5] During the three-month period, households may miss attractive opportunities from the private sector.
In total, this example motivates the three frictions from the model: the private information on priority scores, the truncation on the allocation mechanism and the application cost (modelling the opportunity cost of waiting).

The London social housing allocation scheme ("Choice Based Lettings Scheme") much resembles the Paris scheme, except for the information. On the dedicated website,[6] the application period runs from each Thursday morning until the following Monday midnight. During this period, as a household applies, she observes her position in a priority ranking of all current applications on her targeted housing unit.
Through a process of trials and withdrawals, it is then possible to recover common knowledge on (the order of) priority scores.

Research question

In this paper, we model private information on priorities, jointly with the other realistic / standard market frictions (truncation, participation cost). We study application behaviors, the

[2] With some exceptions, which we discuss in section §2.

[3] To be translated as "rental advertisements".

[4] `https://teleservices.paris.fr/locannonces`.

[5] The example of social housing is in fact dynamic, with successive rounds on application. An agent who does not apply or fails in a given round is offered the opportunity to apply again in the next round. Appealing to a static model, we certainly miss this aspect. We address this issue in section §6.1 by introducing in the static model an endogenous cost of participation which captures one of the main effects of the dynamic. The cost decreases with the priority, reflecting that high-priority agents keep their high score in successive rounds.

[6] `www.homeconnections.org.uk`.

allocation, and the welfare.
We want to be able to predict the outcome of social housing in Paris, in comparison with London. More generally speaking, we try to build a model capturing the dilemma of “Whether and Where to Apply” and giving insights about any market where there is uncertainty on priorities.

Overview of model

To address this agenda, we follow a standard methodology.

We define a stylized frictional matching market: two-sided, one-to-one, agent-object, with non-transferable utilities. Preferences and priorities are homogeneous, meaning that each object is characterized by a unique objective value, and conversely, each agent is endowed with a single (privately known) priority score. The allocation occurs through a truncated Deferred Acceptance (equivalently, truncated Serial Dictatorship) mechanism, with truncation one. Agents independently and simultaneously decide to apply to one or no object, where an application is costly. Then each object goes to the highest priority agent among the pool of applicants.

We model strategic interactions on the market as a Bayesian Game of incomplete information termed “Application Game”. On the market defined, any participant suffers uncertainty on who else applies and wonders “Whether and Where to Apply?”.[7] Her answer to that question should depend on the other participants’ strategies since any higher priority agent applying to the same object eliminates her chances of getting the object. Thus, in building one’s application strategy, the agent must consider the trade-off between being ambitious, accepting the prospect of competition (targeting high-value objects), or being practical, seeking coordination (targeting under-demanded objects).

Preview of results

The analysis elicits the equilibrium application strategies as defined by the Bayes-Nash equilibria of the Application Game. It finds that in any equilibrium, high-score agents are ambitious, and low-score agents are practical. We fully characterize the symmetric equilibrium, and in particular, we show that it is necessarily interior. We also uncover two salient and somehow puzzling features of the symmetric equilibrium, regarding its structure and its efficiency. One, agents with scores on a continuous support sort into discrete classes (defined as groups of close priorities) where they adopt exactly the same strategy.

We compute the welfare associated with equilibrium outcomes using the two criteria of ex ante and interim expected payoffs, and compare it with the level of welfare achieved without the frictions. We derive implications for market design. Although the frictional market design is sub-optimal for the criterion of ex ante welfare, it maximizes participation, and the inefficiencies associated with the described market design are interim asymmetric. In many instances of the Application Game, we even find that low-score agents are better off in the sub-optimal (private information) design than in the optimal (public information) design. The conclusion is that the frictional design is less efficient but more egalitarian than the friction-less design. Private information mitigates the discriminatory power of the priority score system. This calls for a joint design of the priority score computation rule and the information structure.

Outline of paper

The rest of the paper[8] is structured as follows. Section §2 reviews the related literature. Section §3 models the market and the associated game. Section §4 derives the Bayes-Nash equilibria. Section §5 investigates the welfare. Section §6 proposes and processes three natural model extensions. Section §7 concludes. All proofs are available in appendix §A.

[7] Because the model is static, agents do not wonder *when* to apply.

[8] Throughout this work, we regularly make use of the social housing vocabulary. This semantic choice is for illustration purposes, yet it should not conceal the wider ambition of this work, shedding light on any market featuring cardinal priorities.

2 Literature review

Markets with private information or uncertainty on priorities or competition

We are not the first ones to be interested in matching with private information or uncertainty. Yet, in the majority of existing matching papers, the uncertainty applies to one's own ex-post payoff in the match (preferences). In our paper, the uncertainty applies to the probability of acceptance (priorities). It comes from the fact that participants have little information (number and prior distribution of scores) about the competitors they face.

Thus, the present is closer in the spirit to the literature modeling uncertainty about the competition on the market.
For auction problems, Gleyze and Pernoud [10] focus on the uncertainty about other bidders's preferences and Lauermann and Speit (2019) [15] model uncertainty about the number of bidders.
For matching problems, Roth (1989) [20] models agent-agent market with private information on preferences on both sides: an agent only observes her own utility function and holds a prior distribution over the possible vectors of other agents' utilities. As a motivation, Roth (1989) describes the market for hospital interns, where students do not know how hospitals value them. He studies the revelation game induced by direct mechanisms. He shows that results on dominant and dominated strategies are similar to the standard results from the complete information benchmark. But results on Bayes-Nash equilibria are negative: for any mechanism, there exists some prior distributions for which at least some Bayes-Nash equilibria of the resulting game produce unstable matchings.
Kloosterman and Troyan (2020) [14] show that when preferences are uncertain but correlated, DA is no longer strategy-proof or stable and less informed students are worse off due to a curse of acceptance (being accepted at a school signals that the school's quality is low). They show that priority design (so that any student is guaranteed a safe school) mitigates these issues.
Because in Roth (1989) and Kloosterman and Troyan (2020), the uncertainty is on other agents' preferences, an agent's payoff depends on the types of other players only indirectly through the actions of the players. In the agent-object market with private information on priorities from the present paper, the payoff more generally depends both on the actions and on the types of the other players. There are no reporting issues: by definition, the market designer designs the priority system and perfectly observes the priority scores.

Matching with uncertain priorities

One notable exception is the literature on "optimal portfolio choice", mostly in school choice.

Chade, Lewis and Smith (2014) [5] consider a decentralized Bayesian game of admissions gathering two colleges and many heterogeneous students.[9] Colleges have the same value to all students, and each student is characterized by a unique score, so that preferences and priorities are homogeneous. There is a cost of application for students, and in addition, colleges' evaluations of students' applications are uncertain.[10] A student designs her application strategy[11] (no application, application to one college, to both colleges) maximizing her expected payoff, which is the college's value she is admitted to minus application costs. A college designs the admission standard to maximize the total score of its student body under capacity constraints. Their model differs from ours in two ways. They model strategic interactions within the college side and between students and colleges, whereas we are interested in strategic interactions within the agent side. The uncertainty also differs. In their model, there is common knowledge of students' priority types but exogenous noise on the allocation. In our model, uncertainty on priority order endogenously

[9] This model is a special case of the problem of simultaneous search by Chade and Smith (2006) [6].

[10] The key assumption about this uncertainty is a monotone likelihood property for the distribution of signals on students' scores to colleges. Therefore, a higher score student always sends a higher proportion of good signals (vs. bad signals) on her score than a lower score student, so that colleges use cutoff strategies at equilibrium.

[11] The paper-specific terminology says that students make "portfolio choices".

arises from private information on priority scores. Their analysis finds that at Bayes-Nash equilibrium, student-college sorting may fail in two ways: first, weaker students sometimes apply more aggressively; second, weaker colleges might impose higher standards. Our analysis (see section §4.4) finds the opposite and more standard pattern for our agent-object market: at the Bayes-Nash equilibria of the Application Game, higher-value objects are played more often, at higher scores.

Ali and Shorrer (2021) [1] define the general decision problem of students who are uncertain about their (correlated) priorities (whereas our paper studies a game between participants). Their focus is on the correlation between admission chances and the subsequent signaling effects. Because priorities so admissions decisions are correlated, the optimal portfolio involves applying to a combination of “reach”, “match”, and “safety” schools. In our model, as in many applications beyond school choice, priorities are homogeneous (perfectly correlated). We could generate results similar to Ali and Shorrer (2021) only after enlarging the truncation on the mechanism.

Avery and Levin (2010) [2] model students who are differentiated in their academic ability and in their fit for different schools. Each student knows her ability only imperfectly, thus is uncertain about the priority order. The focus of the paper is on early admissions. They show that early admissions have a sorting effect (early applications convey a signal of good fit from students to schools) and a competitive effect (lower-ranked colleges attract cautious high-ranked students).

Frictional matching

This paper more generally relates to the literature on frictional matching, as it models a truncated sub-optimal mechanism and costly application. The ”Whether and Where to Apply” dilemma stems from the fact that truncating the Deferred Acceptance mechanism sacrifices the strategy-proofness (Haeringer and Klijn (2009) [11]).[12]
The novelty in our approach comes from the fact that frictions add up and interplay to create novel strategic interactions arising within the agent side of the market. It combines some aspects of centralized matching (coordination in the timing of application) with some decentralized aspects (private information). The policy recommendation stemming from the welfare analysis is straightforward and should be applied broadly.

Multi-item auction

Because it features homogeneous preferences and priorities, the coming model also bears some similarities with a multi-item auction. In Demange, Gale and Sotomayor (1986) [8], a collection of items is to be distributed among several bidders. All bidders rank items in the same way, and each bidder is to receive at most one item. The truncated Deferred Acceptance mechanism in our matching model and the generalized first-price or second-price mechanisms in the auction model have in common that they fail to be strategy-proof and that they result in the same allocation (when bids in the auction are consistent with priority scores in matching).

[12] The profitable preference manipulation is to include “safe schools” in the rank-ordered list.

3 Model

3.1 Frictional market and Application Game

We consider a market with n agents and m objects $(m \leq n)$. Agents (resp. objects) are numbered by $i \in \{1, \ldots, n\}$ $(j \in \{1, \ldots, m\})$.[13] Agents have common cardinal preferences over objects: all agents assign the same objective value to each object, denoted a^j for object $j \in \{1, \ldots, m\}$. The convention is set that object 1 (resp. m) is the highest (lowest) value object: $a^m < \ldots < a^1$. Objects' preferences over agents (equivalently priorities) are also cardinally common: each agent i is characterized by a unique priority score, denoted ω_i. Priority scores are independently and identically distributed according to some (cumulative) distribution F on the unit interval: $\omega_i \sim F([0,1]),\ i \in \{1, \ldots, n\}$.
Information about priority is private: an agent i only knows her own priority score ω_i, but is ignorant of the priority scores ω_j of other agents $j \neq i$.
The allocation occurs through a truncated Deferred Acceptance mechanism, with truncation one. In this mechanism, agents are asked to independently and simultaneously choose whether to apply to an object and, if yes, to which object. Because priorities are common, the mechanism is equivalent to a serial dictatorship where the serial order is given by the score.
Application is costly, it costs $c < a^m$. If a given object receives no application, then it is wasted. If an object receives exactly one application, it goes to the single applicant. If an object receives at least two applications (crowding), the mechanism selects the agent with the highest priority score among the pool of applicants and endows this agent with the object.[14] In this latter crowding case, we say that the agent who gets the object succeeds, while the other applicants fail.
A successful agent receives the value of the object she is assigned minus the application cost. An agent who has failed just pays the application cost, hence a negative utility. An agent who has chosen not to apply secures a reservation utility of zero.

This model poses a symmetric Bayesian Game of incomplete information that we call "Application Game" (AG). This game comprises n players, with action space $A_i = \{A^1, \ldots, A^m, N\}$ - where A^j denotes the action of applying to object j and N[15] stands for the action of not applying -, privately known independent types, prior F over $[0,1]$, and payoffs:

$$u_i(X_i, X_{-i}) = \begin{cases} a^j - c & \text{if } X_i = A^j \text{ and } \{l \in \{1,\ldots,n\} \backslash \{i\} | X_l = A^j,\ \omega_l > \omega_i\} = \emptyset,\ j \in \{1,\ldots,m\} \\ -c & \text{if } X_i = A^j \text{ and } \{l \in \{1,\ldots,n\} \backslash \{i\} | X_l = A^j,\ \omega_l > \omega_i\} \neq \emptyset,\ j \in \{1,\ldots,m\} \\ 0 & \text{if } X_i = N \end{cases}$$

where $u_i(X_i, X_{-i})$ denotes the payoff of player i when she plays action X_i and the rest of agents play according to action profile X_{-i}.

A pure strategy $s : [0,1] \rightarrow \{A^1, \ldots, A^m, N\}$ in the AG is a mapping from the interval of scores to the set of available actions (distributions over actions). A mixed or behavioral strategy $p : [0,1] \rightarrow \triangle\{A^1, \ldots, A^m, N\}$ is a mapping from the support of scores into the simplex of the action set: $p : [0,1] \rightarrow \triangle\{A^1, \ldots, A^m, N\})$. The probability $p_i^j(\omega),\ j \in \{1, \ldots, m+1\}$ stands for the odds that agent i chooses action A^j when her score is ω. A strategy is interior whenever the probability distribution is non-degenerate on more than a finite number of points, that is when $\exists\ \omega' < \omega'' \in [0,1],\ j \in \{1, \ldots, m+1\}$ s.t. $\forall\ \omega \in [\omega', \omega] :\ 0 < p_i^j(\omega) < 1$.
A strategy can easily be represented graphically, displaying areas of scores where the agent chooses each action; in one dimension ($[0,1]$ line) for pure strategies, two dimensions ($[0,1]$ square) for behavioral strategies.

[13] Notation: In all the following, numerals for agents (resp. objects) are written in indices (exponents).

[14] The mechanism does not specify how to break ties in case crowding happens between several agents with the same priority scores. But since F is a continuous probability distribution, ties occur with probability 0, hence no consequence on payoffs and equilibrium behaviors.

[15] Occasionally, to ease the notations in the rest of the analysis, we denote the action N as an additional application action, $A^{m+1} := N$, and define $a^{m+1} := c$.

3.2 Benchmark Market with perfect information

Throughout the analysis, we continuously refer to an alternative market design with public information. More precisely, priority scores are common knowledge.[16] We call this design the "friction-less" or "benchmark" market.[17] It defines a game with perfect information called "Sorting Game".

The following table summarizes the difference between the frictional and benchmark markets.

Design	Game	Information on priorities
Frictional market	Application Game	Private (perfect observation of own score)
Benchmark market	Sorting Game	Public (common knowledge of scores)

FIGURE 2: Summary of two market designs

The comparison between the two designs, both in terms of predicted behaviors and subsequent welfare, will shed light on the most interesting features of the frictional market.
By definition, the benchmark frictionless market is efficient, whereas the frictional market is inefficient. In section §5, we precisely characterize the inefficiencies associated with the frictions. The general idea is that providing public information on priorities, removing the truncation, or making the mechanism sequential would enable to capture the maximum welfare.

3.3 Model justifications

Modelling a sub-optimal mechanism follows from the observation that market operators most often stick to the frictional design. Paris social housing is just one example. About the truncation, Pathak (2016) [19] observes that in school choice, the truncation is more often the rule than the exception. We propose two families of explanations: constraints and hidden objectives.
On the constraint side, Pathak (2016) suspects that truncation is used because it saves on operational costs that are usually unmodelled in theoretical matching papers. In school choice, reviewing students records takes time. In social housing, organizing viewings of the accommodations also takes a lot of time. Universities and social landlords could try to limit the number of applications they receive to save on this time. With respect to private information, Roth and Sotomayor (1990) [21] note that many two-sided matching markets - in particular, entry-level labor markets - use decentralized application procedures, where agents from the same side of the market are isolated from each other, and information on preferences or priorities is subsequently private. It could also be that public information cannot be achieved due to privacy concerns. Finally, sequentiality is not a solution if there is a high opportunity cost of time (in social housing, this corresponds to the cost of vacancy).
With respect to social objectives, we argue that market operators may have different or additional objectives than just maximizing the ex ante aggregate welfare. With respect to efficiency still, they could be interested in using more sophisticated criteria such as the Pareto order (considering that a market design is superior if it leads to a higher payoff for all levels of scores). In addition to efficiency concerns, they could be interested in participation. Indeed, in many social landlords reports, we observe that statistics about high or increasing number of applications received are proudly announced.
In the baseline model, we impose an extreme truncation on the mechanism: agents can apply to at most on object. This is mostly to keep the model simple and tractable. In section §6.2, we show that our results are robust to a larger truncation.

[16] Common knowledge of the order of score is the most parsimonious information structure leading to the same equilibria.

[17] An alternative but equivalent design would consist in making the mechanism sequential rather than static. The market operator would organize the timing of applications by decreasing order of scores so that lower-ranked agents have the opportunity to observe the highest-priority agent assignments before submitting their own applications.

The homogeneous preferences are an extreme version of correlated preferences,[18] and mostly a first approach. In section §6.3, we show that our results are robust to imperfect correlation in the preferences.
Homogeneous priorities (as materialized with a unique priority score for each agent) fit the many contexts where the needs of the object side of the market are similar and are very standard in many public economic applications. For college admissions, the SAT score, as a weighted average of student performance in a number of maths, reading and writing exercises, serves as a measure of a high school student's readiness for any college. In social housing, the priority score, reflecting the emergency of the household's housing need, controls the priority to any vacant housing unit. In the case of teacher allocation to schools, the same priority score applies to any school where a teacher can apply.

Finally, the cost of application models the opportunity cost of time and effort dedicated to the application (reviewing available objects, sending application), sometimes adding up to an objective application or participation fee.

[18]Homogeneous or correlated preferences are problematic empirically, hence interesting theoretically, because they introduce competition within the sides of the matching market.

4 Equilibria

To study behaviors in the AG, we use Bayes-Nash equilibrium (BNE) as equilibrium concept.

4.1 Preliminaries

We make a few preliminary qualitative remarks on the structure of the problem and introduce the formalism.

4.1.1 Existence

We first state existence of an equilibrium in the AG.

Lemma 1. *[Existence]*
There exists a Bayes-Nash equilibrium of the Application Game.

The proof exploits the Bayes-Nash existence theorem for games with finite action space and independent types (potentially infinite type space) by Milgrom and Weber (1985) [18].

4.1.2 Interim expected payoffs

The interim expected payoff of player i under strategy profile p, when her priority score is ω is denoted $\mathbb{E}[u_i(p)|\omega]$. Due to the dictatorship, in the AG, interim payoffs depend on the strategy of other agents through the behaviors of higher score agents, yet are independent of the behaviors of lower score agents. Trivially, the interim payoffs also depend only on the agent's strategy through the agent's behavior at the set score and not at higher or lower scores. Thus in the formalism for interim payoffs, it is enough to specify for p just $p_i(\omega)$ for the agent, and $p_{-i}([\omega,1])$ for the other agents: $\mathbb{E}[u_i(p)|\omega] = \mathbb{E}[u_i(p_i(\omega), p_{-i}([\omega,1]))|\omega]$.
At BNE $p^\star$, we have:

$$\forall\ i \in \{1,\ldots,n\},\ \forall\ \omega \in [0,1]:\ p_i^\star(\omega) \in \operatorname*{arg\,max}_{p_i(\omega)} \mathbb{E}[u_i(p_i(\omega), p_{-i}([\omega,1]))|\omega]$$

In the case where the agent applies with full probability to one object at score ω, the interim payoff is a weighted sum of two ex-post payoffs: the (positive) object value and the (negative) application cost, where the value is weighted by the conditional probability of success, denoted $\mathbb{P}(S|p,\omega)$:

$$\mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|\omega] = \mathbb{P}(S|p,\omega)\,a^j - c$$

The following lemma characterizes this interim expected payoff:

Lemma 2. *[Interim payoff after pure action - Characterization]*

(i) *Continuity: Interim payoff conditional on any action is continuous in the score.*

(ii) *Monotonicity:*
- *Interim payoff conditional on not applying is constant equal to* 0*.*
- *Interim payoff conditional on any application action is increasing (constant) on any interval of scores where at least one other agent (no other agent) applies to the same object with a positive probability.*

(iii) *Value at highest bound: Interim payoff at score 1 conditional on applying to any object is equal to the value of the object minus the cost.*

Statement (i) states that there is no jump in the interim payoff: one's chance to get an object, hence one's payoff cannot dramatically change from one score to a neighbor score. Statement (ii) says that one's payoff increases when one's score rises if and only if the agent was facing crowding on the targeted object by agents with scores slightly above. Statement (iii) formalizes that the

highest score agent is successful for sure.
The three statements are direct consequences of the serial dictatorship mechanism and the continuity of the priority score support.

4.1.3 Gross incentive analysis

It is well known that once truncated, the Deferred Acceptance mechanism is no longer strategy-proof (Haeringer and Klijn (2009) [11]). The typical preference manipulation consists in applying to “safe” objects (objects which accept the agent with high probabilities). It materializes in different ways at different levels of scores.

A player with a score close to 1 should feel confident that when applying to any object and in case of crowding, she will succeed. She should target high-value objects, fully accepting the prospect of competition.
Conversely, a player with a score approaching 0 should expect that when applying to any object and in case of crowding, she will fail. This low-score agent seeks to avoid competition and to coordinate with peers so as to target different objects: she should target under-demanded objects. Even more than that, she may be tempted not to apply to guarantee a utility of zero. Any agent with an intermediate score faces a trade-off between applying to high-value objects and risking failure or to under-demanded objects (or even giving up) and settling for a low (zero) satisfaction. We illustrate this discussion on the graph below.

FIGURE 3: Coordination and competition behaviors in the AG as a function of priority scores

4.2 Toy examples

As a prelude for the general results, we display the equilibria in an example where dimensions are small ($n = 3 > m = 2$) and the distribution is uniform ($F \sim \mathcal{U}([0,1])$). We provide a graphical representation, a description, and the intuition.

4.2.1 Pure (asymmetric) equilibrium

On the graph below,[19] each line going from 0 to 1 stands for the score support $[0,1]$, one line for each strategy of the three players, and the letters above stand for the action played at the corresponding scores.
The bracket “robust profile” specifies the part of the profile that is realized at the pure strategy equilibrium of any AG.

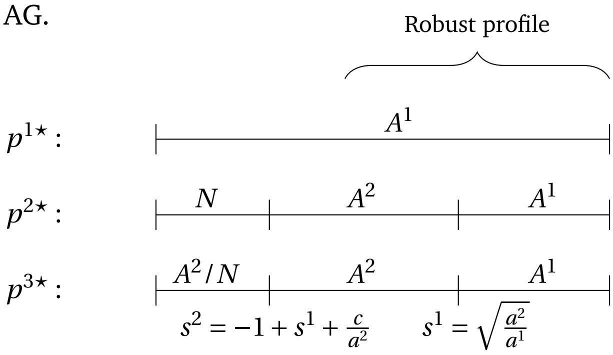


FIGURE 4: Pure (asymmetric) BNE - $n = 3$, $m = 2$, $F \sim \mathcal{U}$

[19] We deal with the asymmetry by numbering players and assigning each of them to a specific role. Yet any permutation of strategies between players is again an equilibrium.

The graph shows three intervals, where the intervals' bounds s^2, s^1 are indifference points.

On an interval of high scores $[s^1, 1]$, all agents apply to object 1. This is because an agent with the highest possible score 1 always gets the object she has applied to. So she applies to the highest value object and secures the highest possible payoff in the game $a^1 - c$.

At score s^1, confidence in success in case of crowding when applying to object 1 becomes quite low. By contrast, the probability of success conditional on applying to object 2 is constant equal to 1. Algebraically, the interim payoff of applying to object 1 hits the value of object 2 minus cost $a^2 - c$. At s^1, agents become indifferent between applying to objects 1 or 2.
On an interval of intermediate scores $[s^2, s^1]$, two agents (agents 2 and 3) apply to object 2. Because they compete, their interim payoff steadily decreases from s^1 leftward. The other agent (agent 1) keeps on applying to object 1. From her point of view, there is no competition anymore on object 1, her interim payoff is constant on the whole interval $[s^2, s^1]$.
The fact that agents share roles (with a majority of applicants to object 2, a minority of applicants to object 1) breaks the possibility of a symmetric equilibrium in pure strategies. To get an intuition on the necessity of asymmetry, we can consider (by contradiction) a symmetric strategy profile where all players would shift to apply to object 2 below s^1. Then, the interim payoff conditional on playing object 1 would be constant (no competition) as the score decreases below s^1, whereas the interim payoff conditional on playing object 2 would increase (2 competitors). Consequently, the former would be higher than the latter at any score below s^1, and any player would face a profitable deviation from object 1 to object 2.
At score s^2, the interim payoff of agents 2 and 3 hit the zero bound. It becomes profitable for one of them (say agent 2) to deviate to N, to secure a payoff of zero. Conditional on that, agent 3 is indifferent between maintaining action A^2 with no competition or playing action N (both deliver the same zero payoff). Agent 1 keeps on playing object 1 on the whole interval $[s^2, s^1]$. She still faces no competition on object 1, hence a constant interim payoff.

The bracket "robust profile" shows that only the right side of the graph is realized at the pure strategy equilibrium of any AG. For some sets of parameters, we may observe only the top part of the profile, and not the bottom part. Agents with score 0 may all apply (potentially only to the two best objects).

4.2.2 Symmetric (interior) equilibrium

On the following graph, the horizontal line in the square still represents the score support $[0, 1]$. The vertical line represent the probabilities of each action in the behavioral strategy.
For example, $p^2_{[\![1,2]\!]}$ denotes the probability with which an intermediate score agent applies to object 2.

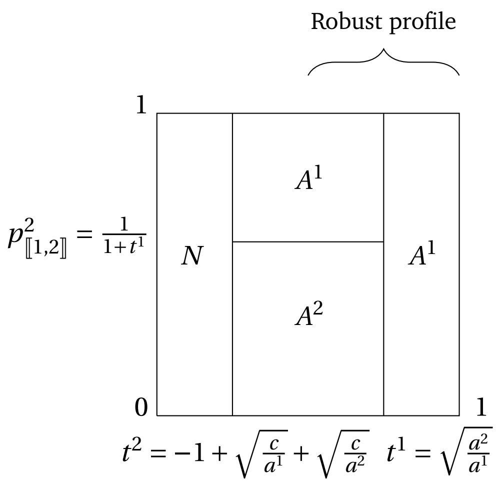


FIGURE 5: Symmetric (interior) BNE - $n = 3,\ m = 2,\ F \sim \mathcal{U}$

As in the pure strategy equilibrium, all agents apply to object 1 at scores belonging to an interval $[t^1, 1]$.
The equilibrium strategy becomes truly interior on an interval $[t^2, t^1]$, where an agent applies with positive probabilities to objects 1 or 2. What is striking here is that those probabilities are constant. This is what we call the "block structure of the equilibrium". In section §4.3.2, we discuss this central result.
Notably, intermediate score agents apply more often to object 2 than to object 1 ($p^2_{[\![1,2]\!]} > \frac{1}{2}$).
At some low score t^2, due to competition by higher-ranked agents, the interim payoff of playing A^1 and A^2 hits the zero bound. Agents choose not to apply at any score below. Symmetry does not allow one player to keep on applying, unlike what happens in the pure case.

The bracket "robust profile" shows that for some parameters of the AG, all agents apply with probability one, (potentially only randomizing on the two best objects).

4.2.3 Summary results from toy example

In both the pure and interior equilibria, confidence in success makes high-priority agents ambitious. At intermediate scores, it becomes rewarding to be less ambitious (say "practical") and to try to coordinate to avoid competition.
In the pure equilibrium, this happens through a sharing of roles between applicants to different objects; in the interior equilibrium, by positive probabilities to apply to both objects. Many intermediate score agents settle for the secure option. A remaining smaller group of intermediate score agents take advantage of alleviated competition to maintain high ambitions. At the lowest possible scores, agents may need to shift to the no application action so as to secure a positive payoff.

Although the structure of the equilibrium is very robust, whether the possibility to abstain or to apply to low value objects is used at equilibrium depends finely on the parameters of the AG.

4.3 General results

The analysis from the toy example generalizes to any number of agents n, objects m, and any priority score distribution F.

4.3.0 Nash equilibrium of the Sorting Game

Proposition 0. *[(Unique) NE]*
In the Sorting Game, there is a unique Nash equilibrium $\sigma^\star$, where:

(i) The agent ranked i^{th}, $2 \le i \le m$ in priority applies to and is allocated object ranked i^{th} in value.

(ii) Agents ranked i^{th}, $m+1 \le i \le n$ in priority do not participate.

The proof is done by induction, following the priority order for agents, which is also the serial order used in the dictatorship mechanism.

On the benchmark market, agents are able to perfectly tailor their ambitions to their ranks in the priority order and endogenously sort. The highest priority agent (second-highest priority agent) applies to the highest value object (second-highest value object), and so on. If there are strictly more agents than objects, the lowest score agents do not apply to avoid certain failure. In total, each available object receives exactly one application, which is accepted. No object is wasted and no agent fails.

4.3.1 Pure (asymmetric) Bayes-Nash equilibrium of the Application Game

Proposition 1. *[Asymmetry of pure BNE]*
A pure strategy Bayes-Nash equilibrium of the Application Game is necessarily asymmetric.

The proof is done by contradiction, just as in the toy example.

The short intuition is that competition by high-score agents on the highest-value objects smooths interim payoffs conditional on different application actions. Consequently, at any intermediate or low score, several objects of different values are equally attractive. To guarantee the absence of profitable deviation, they all need to be targeted by at least one agent with positive probability. A pure symmetric profile would not permit that.

In the rest of the paper, we discard the BNE in pure strategies, for two reasons. First, we remain skeptical about the capacity of (ex ante symmetric) agents to coordinate to share the different roles in an asymmetric profile, absent any communication. Second, the pure strategy BNE structure is little robust. In the general model with any number of agents, objects, and any distribution, it depends very finely on the set of parameters of the game. A general characterization would be very tedious to write. We illustrate this lack of robustness in appendix §B.

4.3.2 Symmetric (interior) Bayes-Nash equilibrium of the Application Game

The following theorem states existence and uniqueness of a symmetric BNE and describes a very specific equilibrium structure.

Theorem 1. *[Symmetric (interior) BNE]*
A symmetric (interior) Bayes-Nash equilibrium $p^\star$ of the Application Game:

(1) Exists and is unique.

(2) Exhibits a "block structure", meaning that there is a finite number of intervals of scores, called "classes" where the interim strategy profile at any score is the same:

(i) There are between 2 and $m+1$ classes: $k_0(p^\star) \in \{2,\dots,m+1\}$.
More precisely, there are exactly:

- k, $k \in \{2,\dots,m\}$ *classes iff:* $1 + \sum_{l=1}^{k} \left(\frac{a^{k+1}}{a^l}\right)^{\frac{1}{n-1}} \leq k < 2 + \sum_{l=1}^{k-1} \left(\frac{a^k}{a^l}\right)^{\frac{1}{n-1}}$
- $m+1$ *classes iff:* $m < 1 + \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}}$

(ii) Conditional on existence, classes write $[t^k, t^{k-1}]$, with:

$$\forall\ k \in \{1,\dots,k_0(p^\star)-1\}:\ t^k = F^{-1}\Big(1 - k + \sum_{l=1}^{k} \left(\frac{a^{k+1}}{a^l}\right)^{\frac{1}{n-1}}\Big)$$

$$t^{k_0(p^\star)} = 0$$

(iii) Conditional on existence:

- *In class $k \in \{1,\dots,m\}$, agents apply to object $j \in \{1,\dots,k\}$, with probability:*

$$p^j_{[\![1,k]\!]} := \Big(\sum_{l=1}^{k} \left(\frac{a^j}{a^l}\right)^{\frac{1}{n-1}}\Big)^{-1}$$

- *In class $m+1$, agents do not apply.*

The next figure illustrates the symmetric equilibrium.

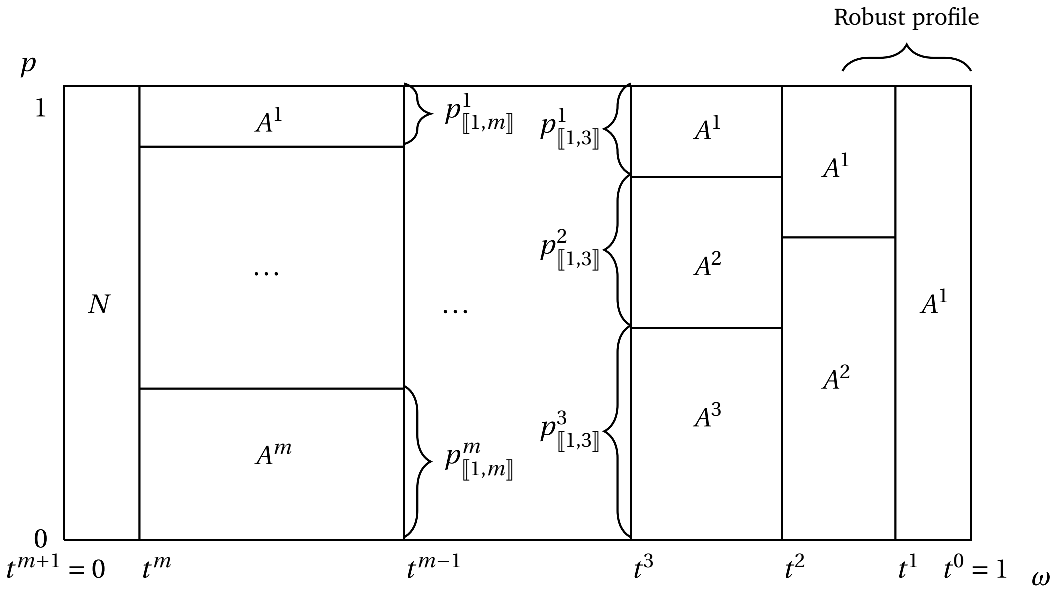


FIGURE 6: Symmetric (interior) BNE of the AG – General form

From now on, the bounds of the classes t^k are "thresholds", and the constant probabilities $p^j_{[\![1,k]\!]}$ are "levels".[20]

The proof of theorem 1. is done in three steps.
In the first step, we prove that we can divide the score support $[0,1]$ in a finite number of intervals where agents with scores in one interval apply with positive probabilities to only object 1, then both object 1 and 2 until an interval where they potentially apply to all objects and a bottom interval with no application. This step relies heavily on lemma 2., jointly with the intermediate value theorem applied recursively m times. The proof that t^1 necessarily exists is made by contradiction, exactly as in the sketched proof of proposition 1..
The next step is to characterize the probability functions. The proof that they are piece-wise constant is done by induction. At inductive step $k \in \{2,\dots,m-1\}$, the strong indifference principle applied at a score $\omega^\star \in (t^k, t^{k-1})$ delivers a system of $k-1$ differential equations with $k-1$ unknowns $(p^j(\omega^\star), j \in \{2,\dots,k\})$. Substituting within the equations, we find a relation between the primitives of $f p^j$ and f, hence constant probabilities $\forall\ \omega \in [t^k, t^{k-1}],\ p^j(\omega) := p^j_{[\![1,k]\!]}$. Meanwhile, we use the differential equations again to get a recursive relation between all $p^j_{[\![1,k]\!]}$, $j \in \{1,\dots,k\}$, and use the fact that they sum up to one to get the explicit formula.
In the third and final step, we use the thresholds' definitions (t^k, $k \in \{1,\dots,m\}$ is the highest score where there is indifference between all actions $A^1,\dots,A^{k+1}$) to derive their expressions.

The symmetric equilibrium in small (§4.2) and general dimensions displays the same pattern, and the overall interpretation (intermediary between cooperation game at low scores and conflict game at high scores) is similar. However, the general version sheds light on two interesting equilibrium features deserving a dedicated discussion, in the coming paragraphs.

4.3.3 Block structure and sorting

This block structure of the symmetric equilibrium may appear surprising at first glance.
From a theoretical perspective, it says that within a very large strategy space made of potentially uneven probability functions, agents effectively use a small number of application mixtures. Starting from a continuous type support, we end up with a discrete number of equilibrium behaviors.

[20] Appendix §B.2 provides comparative statics showing how the thresholds and levels vary with the parameters. In particular, it shows that when the distribution of scores change, the levels remain unchanged, only the width of the classes adapt, so that the expected mass of agent belonging to each class remains constant.

From an applied perspective, it means that agents self-sort according to their priority scores into a finite and quite small number of classes. Two households with close priority scores (in the same class) use exactly the same (potentially highly sophisticated) application mixture.
We could have rather expected that higher priority agents would be strictly more ambitious than lower priority agents.

The algebraic necessity of the block structure is clear enough. It comes from the following observation (made on class $[t^2, t^1]$, to fix ideas). The indifference principle, when applied at a score slightly below threshold t^1 (say $t^1 - \epsilon$) when indifference at t^1 is already established, delivers exactly the same constraint on the probability function as when applied slightly leftward at a score $t^1 - 2\epsilon$, when indifference at $t^1 - \epsilon$ is already established. This constraint writes as the ratio of probabilities of success conditional on applying to different objects is equal to a constant. In both cases, it is thus expected to deliver the same level $p^2_{[\![1,2]\!]}$.

A more direct intuition combines the notions of ambition and risk. The class defines the strategy, hence a constant level of ambition in each class. By contrast, the risk (as measured by the probability of failure) varies within a given class: it is high (low) just above (below) the thresholds. Therefore, for agents at the bottom of a class, the equilibrium strategy features a given level of ambition and is risky; hence a low payoff. For agents at the top of the same class, the equilibrium strategy features the same level of ambition but a low risk; hence a high payoff. The risk and payoff variations are smooth in between. We illustrate this line of intuition below with a figure eliciting the variations of risk and ambition within and between classes.

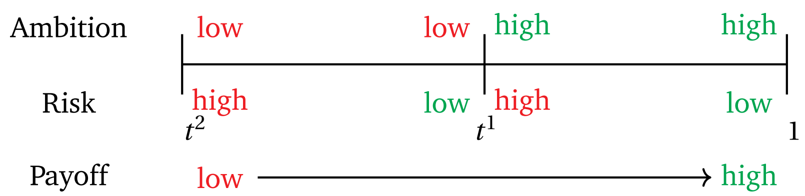


FIGURE 7: ambition, risk and interim payoffs at symmetric BNE of the AG

In total, the block structure is reminiscent of the "class segregation result" in the dynamic search problems. The dynamic search literature (McNamara and Collins (1990) [16], Burdett and Coles (1997) [4], Bloch and Ryder (2000) [3], Jacquet and Tan (2007) [12]) studies two-sided agent-agent markets where each agent is characterized by a value distributed on a continuous support. At each time period, agents are tentatively matched, they observe each other's values and decide to accept or reject the proposed match. At equilibrium, agents sort into a finite number of classes (value intervals) where all agents use exactly the same acceptance cutoff and match within classes. In their case, the equilibrium with class segregation is in pure strategies. In our case, the block structure is even more surprising as agents use sophisticated behavioral strategies.

A related interesting question is whether, at equilibrium, agents are able to self-sort, conforming their ambitions to their scores. It matters in relation to the frictionless benchmark, where the equilibrium outcome is the perfectly positive assortative matching. In our environment with private information, truncation, and application cost, we would like to know whether some assortativity remains. The following corollary answers in a positive way. There is a kind of sorting, although, by the block structure, it proceeds with discrete jumps.

Corollary 1. *[Sorting at symmetric BNE]*
At the unique symmetric (interior) Bayes-Nash equilibrium of the Application Game:

(i) For a given object $j \in \{1, \dots, m\}$ and two scores $0 \le \omega < \omega' \le 1$ where the object receives applications, it receives more applications at the higher score ω'.

(ii) At any two scores $0 \le \omega < \omega' \le 1$, the probability level vector at the higher score $p(\omega')$ first-order stochastically dominates the same vector at the lower score $p(\omega)$.

(iii) *For a given score $0 \le \omega \le 1$ and two objects $1 \le j < j' \le m$ that receive applications at this score, the lower value object j' receives strictly more applications.*

(iv) *In total, conditional on existence of class m, the ex ante probability to apply to object k is $1-\left(\frac{c}{a^k}\right)^{\frac{1}{n-1}}$.*

Statement (i) states that as the score increases, agents apply more and more frequently to high-value objects.
Statement (iii) uses the standard criterion of First Order Stochastic Dominance (FOSD).[21] What it means is that the probability of playing the highest value object is increasing in the score, the cumulative probability of playing one of the two highest value objects is also increasing in the score, and so forth.
Interestingly, statement (iii) also claims that within class $[t^2, t^1]$ (for example), agents apply more often to object 2 than to object 1 (as displayed on FIGURE 6). Hence, the equilibrium appears closer to a pure strategy equilibrium with perfect vertical sorting than to a single-class block equilibrium with no sorting.
Statement (iv) computes the ex ante probability with which an agent applies to a given object, giving a sense of how bankable an object is. It finds that it is increasing with the value of the object a^k (higher-value objects are played more often), decreasing with the application cost c, independent of others objects' values a^j, $j \neq k$ and of the score distribution.

4.3.4 Robust profile and participation

Theorem 1. $(2)(i)$ ensures that the two highest classes $[t^1, 1]$ and $[t^2, t^1]$ are realized at symmetric equilibrium of any AG. Yet, it does not guarantee that any lower class is reached. For some sets of parameters (low cost, low and balanced number of objects and agents, heterogeneous values), classes 3 to $m+1$ (interval $[0, t^2]$) may not be observed ($t^2 = 0$), and low score agents may apply to objects 1 and 2 only. This is the meaning of the vocabulary "robust profile" for $[t^2, 1]$.
In particular, in the extreme case with only 2 classes, this suggests that agents with very different scores (distant from about a half) use exactly the same randomization over actions.

The robust profile bracket also characterizes participation in the mechanism. We say that an agent (fully) participates if she applies with positive (one) probability. The next corollary discusses participation in the AG.

Corollary 2.
At the unique symmetric (interior) Bayes-Nash equilibrium of the Application Game:

(i) *If $m \ge 1 + \sum_{l=1}^{m} \left(\frac{c}{a^j}\right)^{\frac{1}{n-1}}$, all agents on the market participate.*

(ii) *If $m \ge \frac{n}{n-1} \sum_{l=1}^{m} \left(\frac{c}{a^j}\right)^{\frac{1}{n-1}}$, expected participation is higher than in the Nash equilibrium of the Sorting Game.*

Statement (i) states that whenever class $[0, t^m]$ is not reached, all agents on the market - including the ones with the lowest possible scores-, fully participate. They do so in spite of the congestion (the fact that there are fewer objects than agents on the market). This is a major contrast with the benchmark design, where the agents with the lowest possible scores do not apply.
The more general statement (ii) gives a sufficient condition for higher participation in the frictional market. The condition is easily satisfied, as soon as n is not too small or values are quite heterogeneous, or the cost is low.
The lesson is that frictions enhance participation. This has major welfare consequences, which we explore in the next section §5.

[21] First-order stochastic dominance translated to our (discrete) case is given by: $p(\omega) \succeq_{FOSD} p(\omega')$ if $\forall\ j \in \{1, \ldots, m-1\}$, $\sum_{l=1}^{j} p^j(\omega) \ge \sum_{l=1}^{j} p^j(\omega')$.

5 Welfare

In this section, we compute the welfare at the symmetric equilibrium of the AG (§4.3.2). We compare this welfare from the fictional market to the welfare on the benchmark frictionless market.

5.1 Ex ante

We first examine the ex ante payoff, that is, for an anonymous agent, before her priority score is realized. It is a measure of aggregate welfare.

Inefficiencies

The ex ante payoff on the benchmark market is given by:

$$W^B := \mathbb{E}[u(\sigma^*)] = \big(\sum_{k=1}^{m} a^k\big) - cm$$

There is no waste (agents collectively capture the whole sum of objects' values) and no failure (the number of agents paying the cost equals the number of applicants equals the number of objects (m)).

The ex ante expected payoff on the frictional market[22] is given by:

$$W^F := \mathbb{E}[u(p^*)] == \frac{1}{n}\big(\sum_{k=1}^{m} a^k\big) - \frac{c}{n}\Big(m - (n-1)\sum_{j=1}^{m}\big(\frac{c}{a^j}\big)^{\frac{1}{n-1}}\Big)$$

Therefore, the difference in welfare is:

$$W^B - W^F := c(n-1)\Big(m - \sum_{j=1}^{m}\big(\frac{c}{a^j}\big)^{\frac{1}{n-1}}\Big)$$

Proposition 2. *[ex ante welfare]*

(B) The benchmark market design is efficient.

(F) The frictional market is inefficient: $W^B - W^F > 0$.
The size of the inefficiency increases with the values of all objects and with the cost of application.

(B) recalls that, by definition, the friction-less design achieves a higher total welfare than the frictional design. The comparative statics in (F) is straightforward. When an object's value increases, waste on this object is more detrimental to welfare. When the application cost increases, failure is also more consequential.

5.2 Interim

We push the characterization of the inefficiencies identified in section §5.1. We are not only interested in the size, but also in the shape of the inefficiencies.

5.2.1 Interim expected payoffs

The interim expected equilibrium payoff in the benchmark market is given by:

$$W^B(\omega) := \mathbb{E}[u(\sigma^\star)|\omega] = \sum_{j=1}^{m}\binom{n-1}{j-1}(1-F(\omega))^{j-1}F(\omega)^{n-j}(a^j - c)$$

[22] The formula is written for the case $t^m \geq 0$. For case $t^m < 0$, the formula is more sophisticated but proposition 2. remains valid.

It is just the sum of probabilities that the agent with score ω is ranked i^{th}, $i \geq m$ in priority multiplied by the ex-post payoff in this case (object value minus application cost). If the agent is ranked lower (i^{th}, $i > m$), the ex-post payoff is zero.

The interim expected equilibrium payoff in the frictional market is given by:

$$W^F(\omega) := \mathbb{E}[u(p^\star)|\omega] = \begin{cases} \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \cdot \left(\left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \right)^{n-1} a^j - c & , \omega \in [t^k, t^{k-1}],\ k \in \{1, \ldots, m\} \\ 0 & , \omega \in [0, t^m] \end{cases}$$

At any score where the agent applies, it is an expected sum. $p^j_{[\![1,k]\!]}$ is the probability that the agent applies to object j, it multiplies the probability of success conditional on applying to object j. To succeed, one needs that each of all other agents $(n-1)$ does not apply to the same object or has a lower score, which happens with probability $\left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]}$.[23]

How the interim expected payoff varies with the score indicates the effect of priority on individual welfare. If the priority score system fulfills the role of discriminating between agents with different levels of priority, the interim expected payoff should increase with the score.

Lemma 3. *[Equilibrium interim payoffs]*

(B) *On the benchmark market, the interim equilibrium payoff is continuous and strictly increasing with priority score ω.*

(F) *On the frictional market, the interim equilibrium payoff is continuous and:*

- *Constant on $[0, t^m]$ ($t^m > 0$).*
- *Strictly increasing on $[t^m, 1]$*

The proof of (B) is from the formula above, plus the continuity and monotonicity of the cdf F. The proof for (F) just stems from the initial characterization of interim payoffs in lemma and the symmetric equilibrium structure from theorem 1..
Overall, in both kinds of markets, higher score agents are always better off. This means the block structure (featuring constant ambition on each class of scores) still allows for continuously and strictly increasing payoffs. For illustration, we display below two graphs for interim expected payoffs on the frictional market, in the context of the toy example from section §4.2: one with $k_0 = 2$ classes, one with $k_0 = 3$ classes.

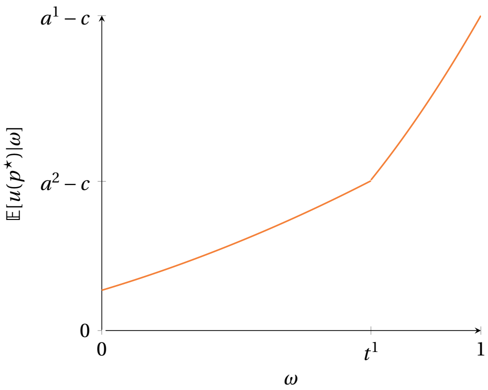


FIGURE 8: Interim expected payoff at the symmetric (interior) BNE - $m = 2,\ n = 3,\ a^2 = 2,\ a^1 = 4,\ c = 0.2,\ F \sim \mathcal{U}$

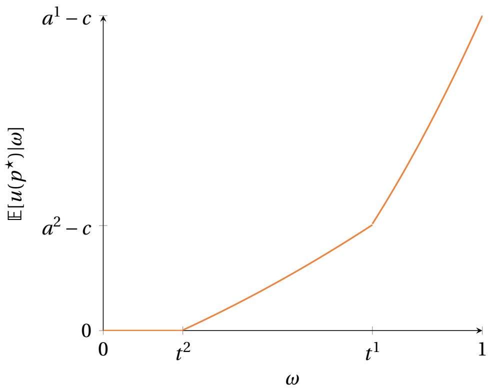


FIGURE 9: Interim expected payoff at the symmetric (interior) BNE - $m = 2,\ n = 3,\ a^2 = 2,\ a^1 = 4,\ c = 1,\ F \sim \mathcal{U}$

[23] The fraction$\left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}}$ simplifies the whole probability of an agent having a score in a higher class and applying to the same object j.

5.2.2 Asymmetries

FIGURE 8 specifically shows that on the frictional market, when $k_0 < m$, agents with low scores apply and expect a positive payoff. In the benchmark market (proposition 0.), by contrast, low-score agents decide not to apply to secure a zero payoff.
This is a clue that there is a score-related asymmetry in the way the inefficiencies associated with private information affect market participants. The next proposition shows that this asymmetry can make low-score agents prefer the situation when everyone has less information.

Proposition 3. *[Low score agents' preference for private information]*
If $m > 1 + \sum_{l=1}^{k} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}}$*, then* $\exists\ \omega' \in (0,1)$ *s.t.:* $\forall \omega \in [0, \omega'),\ W^B(\omega) < W^F(\omega)$.

Thus, in some instances of the AG, low-score agents are better off with less (private) information than with full (public) information. The condition on parameters looks quite general, easily satisfied if the number of objects is large enough and the application cost is reasonably low.[24] One important implication is that we cannot rank the two benchmark and frictional designs through a Pareto order.
The proof uses the definition of t^m as the indifference point between application actions and the no application action, hence an expected payoff of zero, jointly with continuity and strict monotonicity of interim payoff above t^m (lemma 2.). When $t^m < 0$, the monotonicity mechanically induces a positive interim payoff at score zero, to compare with an interim payoff in the benchmark design of 0. The continuity of payoffs extends the comparison to a non-degenerate interval of low scores $[0, \omega')$.
A graphical illustration is given below. FIGURE 10 (left) shows the interim expected payoffs from the frictional market in orange, and the interim expected payoffs from the benchmark market in green. On FIGURE 10 (right) the filled orange area displays the difference in interim payoffs $W^B(\omega) - W^F(\omega)$.

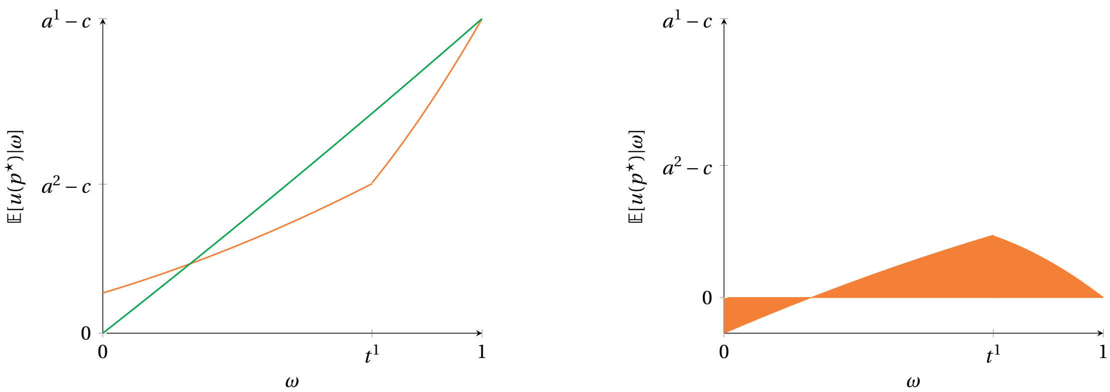


FIGURE 10: Frictional vs. benchmark markets: Interim expected payoffs at symmetric (interior) BNE
$m = 2,\ n = 3,\ a^2 = 2,\ a^1 = 4,\ c = 0.2,\ F \sim \mathcal{U}$

A more general statement from this graph is that intermediate score agents bear most of the burden associated with the inefficiency of private information.

5.2.3 Signal quality, competitive advantage, and competition easing

We formulate the intuition in successive effects of private information on cardinal scores: discrepant signal quality, competitive advantage, and competition easing.

In the application game, although all agents receive signals of the same nature (perfect observation of their own priority score), the informative value of the signal depends on the score level.

[24] Notably, it seems to match our empirical driving empirical motivation of social housing.

A low-score agent, by observing a signal at the bottom of the prior distribution support, realizes that she is almost surely the lowest type, ranked at the bottom of the priority order. By contrast, for an intermediate score agent, observing her score is a poor signal of her ranking. But ranking or ordinality of scores (as opposed to cardinality) is what decides on the allocation in case of crowding. When it comes to information, the low-score agent benefits from a relative competitive advantage. This advantage more than offsets the fact that, in absolute terms, the private information they get is slightly less revealing.
It translates into actions in the following way. The intermediate score agent, misguided by imprecise posterior on ranking, is likely to make mistakes (compared to what she would do with perfect information): being too or too little ambitious, miscoordinating with intermediate-to-high score agents. In total, and due to the recursive structure of the AG, these mistakes tend to alleviate the fierceness of competition by intermediate score agents.
Thanks to the precision of her posterior, the low-score agent avoids any mistake due to unrealistic ambitions. More importantly, she is fully aware of the competition easing. Therefore, she understands that this leaves some room for her to apply. She applies and captures a positive expected payoff.[25]

The result that frictions favor participation and equity is very general and can be reconstructed in many different economic settings. For example, Mekonnen (2019) [17] compares random and directed search on an agent-object market with also common preferences but homogeneous agents. This is equivalent to comparing a no information design to a full information design. At equilibrium, an agent is better off under the random search because she benefits more from the ease of congestion on high-value objects than she suffers from not being able to target objects accurately. Che and Tercieux (2021) [7] study the optimal design of a queueing system when agents' arrival and servicing are governed by a general Markov process. They show the optimal information is no information (beyond recommendation to join / stay in / leave the queue). The intuition is that no information pools the various incentive constraints, ensuring more participation, which, in the queue environment, increases efficiency.

5.2.4 Implications for decentralized matching markets

We have found that in a very stylized frictional market, private information on priorities tends to favor low-score agents.
The question therefore arises: is it an issue for the proper functioning of markets?
The answer to that question rests on the role of priority score systems. In social housing, for instance, the priority system recognizes differential rights to housing based on different levels of emergency. During the allocation stage, it maps those differential rights into proportionally different probabilities of satisfaction. In particular, point rules and allocation mechanisms are often jointly thought of as giving twice more chances to an agent with a score 2ω of getting an object or the best object than to an agent with a score ω.
Yet, in the considered design, private information artificially distorts in an increasing fashion the probability for low-score agents to get a valuable object with respect to intermediate and high-score agents. In other words, private information mitigates the ability of the priority system to discriminate between agents.

From a design perspective, we have found that three elements matter for the final allocation and for welfare: the allocation mechanism, the inputs in the mechanism (the exact rule used to compute priority scores), and the information structure. Our analysis suggests that they should be designed jointly in order to keep control over the amount of discrimination on the market.

[25] A wonder may be: how does private information affect the competition between high-score agents and intermediate-score agents? A high-score agent observes a score close to the higher bound of the support, which is a clear signal that she is likely the highest type, so this agent also benefits from a relative informational competitive advantage. But because she always plays the same action (applying to the best object) in the two designs, she cannot exploit this advantage, and there is no competition easing effect from which the intermediate score agent could benefit.

6 Extensions

The lessons we have learned from our simple model extend to more sophisticated and realistic markets.

6.1 Endogenous cost and the dynamic

In reality, decentralized allocation is always a dynamic process. This is also true in the leading example of social housing in Paris, where each week, a new application round opens and new vacant accomodations become available.
There are two ways to include a dynamic in the model, thereby increasing its descriptive strength. The first and probably most natural way is to model the dynamic explicitly and study the subsequent dynamic search problem.[26] The second and undoubtedly more tractable way is to endogenize some previously exogenous parameters of the static model to display the main effects of the dynamic while remaining in a static and simple framework. In this line of idea, it must be considered that in a dynamic version of the allocation, an agent with a high priority who fails in a given round necessarily keeps a high chance of being allocated an object in future rounds. Her continuation value is high. One way to capture this effect in the static model is to have the cost depend negatively on the priority score. C is now a strictly decreasing function of ω: $c'(\omega) > 0$. We set $c(0) < a^m$ implying $\forall\ \omega \in [0,1], j \in \{1,\ldots,m\}$, $c(\omega) < a^j$ (the cost never exceeds the value of any object).

Proposition 4. *[BNE with score-dependent cost]*
In the application game with score-dependent strictly decreasing cost $c(\omega)$, $c'(\omega) < 0$, a symmetric (interior) Bayes-Nash equilibrium:

(1) Exists and is unique.

(2) Is similar to the symmetric Bayes-Nash equilibrium of the Application Game with exogenous cost:

(i) If $t^m < 0$, the equilibria are exactly the same with exogenous and score-dependent costs.

(ii) If $t^m > 0$, the equilibria are the same except that t^m is higher in with endogenous cost and the m^{th} class $[t^m, t^{m-1}]$ is narrower.

In the proof for the exogenous cost model equilibrium, it was already apparent that thresholds and probability levels in the domain where agents apply with full probability were independent of the cost (with cost functions canceling out on both sides of indifference differential equations). Only the indifference equations between application actions A^j and no application N at t^m feature the application cost on one side.
The implication is that with an endogenous cost rising sharply, the agent applies a little less to all objects by being more prudent at low scores. The lowest value objects suffer the larger decrease in applications.
For illustration, we display the symmetric equilibrium for the small dimensional AG with a score-dependent (linear) cost: $c(\omega) = \frac{3}{2} - \omega$. For comparison, we also display the equilibrium on the same market with exogenous cost $c = 1$:[27]

[26] Stationarity assumptions may help: agents and objects leaving the market are replaced by agents with similar priorities and objects with similar values, and agents staying on the market during several periods keep their priorities.
[27] Note that the expected cost is the same in both cases (equals 1).

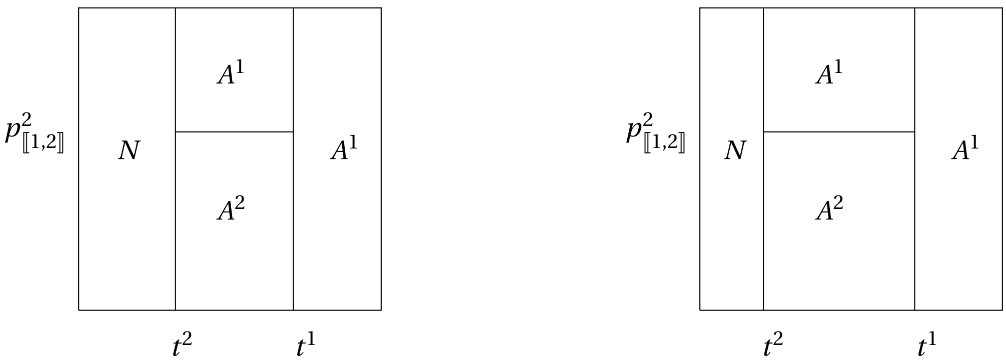


FIGURE 11: Endogenous cost $c(\omega) = \frac{3}{2} - \omega$ (left) vs constant cost $c = 1$ (right)
Symmetric (interior) BNE $m = 2,\ n = 3,\ a^2 = 2,\ a^1 = 4,\ F \sim \mathcal{U}$

Corollary 3. *[Welfare with endogenous cost]*
On the frictional market with score-dependent strictly decreasing cost $c(\omega)$, $c'(\omega) < 0$, the equilibrium interim expected payoff rises faster ($\frac{\partial W^B(\omega)}{\partial \omega}$ larger) on $[t^m, 1]$ than with exogenous cost.

We illustrate this point below:

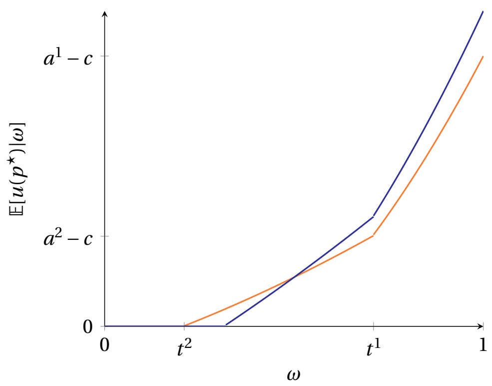


FIGURE 12: Endogenous cost $c(\omega) = \frac{3}{2} - \omega$ (blue) vs constant cost $c = 1$ (orange)
Interim expected payoff at the symmetric (interior) BNE - $m = 2,\ n = 3,\ a^2 = 2,\ a^1 = 4,\ F \sim \mathcal{U}$

In conclusion, endogenous cost induces more discrimination according to score. The market outcome is closer to the outcome of the benchmark market (more discrimination). Accounting for the dynamic will reduce the magnitude of the welfare effects identified in section §5.2.

6.2 Larger or no truncation

In many real-life matching markets, agents are allowed to apply to more than one object. This is usually the case in school choice or centralized job market. In particular, the French national system for allocating teachers to schools uses a priority point system and a mechanism akin to a serial dictatorship where teachers can apply to several schools.[28]
Thus, a natural extension of our model consists in relaxing or removing the truncation of the application menu.[29]
To simplify, we process this extension within the framework of the toy example with $n = 3,\ m = 2,\ F \sim \mathcal{U}$. With no truncation, the action space includes an additional action B for "both" that consists in applying to both objects on the market (hence paying the application cost twice).

Preliminary results are summarized in the next proposition.

[28] In two steps: The first step manages allocation between regions, and the second stage within regions.
[29] Due to homogeneous preferences, the ranking between objects is common, and the action space consists of menus rather than rank-ordered lists.

Proposition 5. *[BNE with no truncation]*
A symmetric (interior) Bayes-Nash equilibrium of the Application Game with $n=3,\ m=2,\ F\sim\mathcal{U}$ and no truncation:

(1) Exists and is unique

(2) (i) If $\frac{a^2}{a^1}+\frac{c}{a^2}>1$, the symmetric equilibrium is the same as with the truncation. In particular, agents always apply to at most one object.

(ii) If $\frac{a^2}{a^1}+\frac{c}{a^2}<1$, at symmetric equilibrium: agents with large scores apply to object 1, agents with scores lower than a threshold r^1 apply to both objects.

Case (i), where agents disregard the possibility of applying to both objects arises when the application cost is high relative to the objects' values. In case (ii), agents use the possibility to apply to all objects, aiming for the high-value object, but hedging against the possibility that it may not be available anymore. Below r^1, all interim expected payoffs strictly decrease due to competition on both objects, and what is the next shift in action is non-obvious (and non-robust).

The figure below illustrates the two cases:

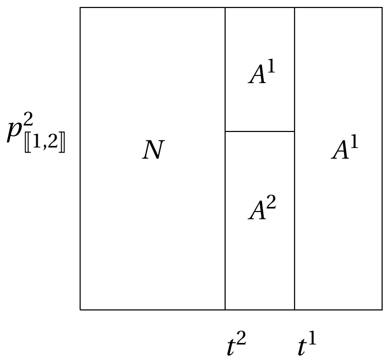


FIGURE 13: BNE with no truncation - Case (i)
$m=2,\ n=3,\ a^2=2,\ a^1=4,\ c=1.5,\ F\sim\mathcal{U}$

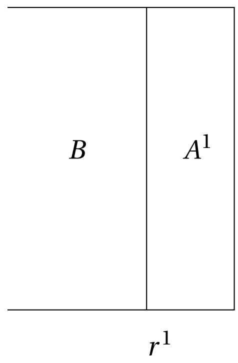


FIGURE 14: BNE with no truncation - Case (ii)
$m=2,\ n=3,\ a^2=2,\ a^1=4,\ c=0.5,\ F\sim\mathcal{U}$

6.3 Imperfectly correlated preferences

The assumption that preferences are homogeneous, with any agent assigning exactly the same value as her peers to any object, is rather restrictive. In social housing, for example, some criteria are valuable to all applicants (size of the accommodation, equipment), but applicants may value (for example) different micro-locations differently due, for example, to the location of their jobs. All in all, individual preferences likely combine common and idiosyncratic components.

We model imperfectly correlated preferences in a simple setting with $n=2$ agents and $m=2$ objects, and a uniform priority distribution. The objects can have two possible values $v>u>0$, so that each agent has exactly one most preferred object with value v and one least preferred object with value u. A preference profile $(X_1X_2),\ X_i\in\{u,v\}$ means that object 1 has value X_1 to agent 1, X_2 to agent 2.
The prior distribution over preference profiles is such that: objects are the same ex ante (they are equally likely to be each agent's most preferred object), but preferences are correlated (positive correlation when $\theta>\frac{1}{2}$):

$$\mathbb{P}(uu)=\mathbb{P}(vv)=\frac{\theta}{2},\ \mathbb{P}(uv)=\mathbb{P}(vu)=\frac{1-\theta}{2},\ \theta\in[0,1]$$

We assume that preferences, just as priority scores, are private information. Thus, a type is two-dimensional: it specifies the priority (score) and the preference (most preferred object), with independence between the two dimensions.

In this setting, a strategy is a mapping of the score support into two possible actions: applying to one's most preferred object (denoted $\oplus$) or to one's least preferred object (denoted $\ominus$).

Proposition 6. *[BNE with correlated preferences]*
A symmetric (interior) Bayes-Nash equilibrium of the Application Game ($n = 2,\ m = 2,\ F \sim \mathcal{U}$):

(1) Exists and is unique.

(2) Has the following block structure:

(i) There are between 1 and 3 classes.

(ii) In the top class $[t^1, 1]$, the agent plays $\oplus$ with full probability, $t^1 = \frac{\theta u-(1-\theta)v}{\theta v+(1-\theta)u}$.

(ii) Conditional on existence, in intermediate class $[t^2, t^1]$, the agent plays $\oplus$ with probability $p(\oplus) = \frac{\theta u-(1-\theta)v}{(2\theta-1)(u+v)}$, $\ominus$ with probability $p(\ominus) = 1 - p(\oplus)$

(iii) Conditional on existence, in the bottom class $[0, t^2]$, the agent does not apply.

At any equilibrium, and as expected, agents with high scores are ambitious and apply to their most preferred object ($\oplus$). It may become more profitable at lower scores (below a score t^1) to also target one's least preferred object ($\ominus$), because, in expectation, this object is less demanded. Interestingly, the block structure remains.
The difference with the perfect correlation case is that the shift at t^1 does not necessarily happen. Indeed, the fact all agents play $\oplus$ combined with the imperfect correlation guarantees that both objects receive applications with positive probabilities. Even when both agents are ambitious, there is partial coordination. Thus, it can be that all agents keep on with the same strategy at low scores. The shift happens if and only if the correlation is sufficiently strong and the gap between the two object values is sufficiently small ($\theta > \frac{v}{v+u}$).

The figure below illustrates the two cases:

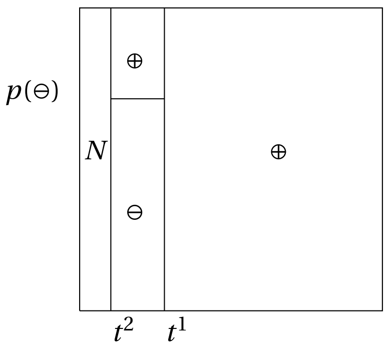


FIGURE 15: BNE with correlated preferences
$m = n = 2,\ F \sim \mathcal{U},\ v = 2,\ u = 1,\ c = 0.5,\ \theta = 0.75$

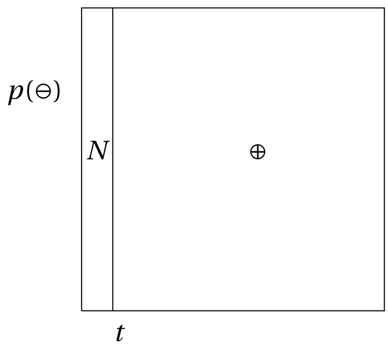


FIGURE 16: BNE with correlated preferences
$m = n = 2,\ F \sim \mathcal{U},\ v = 5,\ u = 4,\ c = 1,\ \theta = 0.75$

7 Conclusion

This paper models a stylized market where agents with homogeneous preferences and privately known priority scores can (costly) apply to at most one object, and each object is assigned to its highest priority applicant. On this market, the frictions (private information, truncation of the mechanism, and application cost) ask for a trade-off between competition and coordination, with participants wondering "Whether and Where to Apply?". They consider the trade-off between being ambitious, accepting the prospect of competition (targeting high-value objects), or being practical, seeking coordination (targeting under-demanded objects).

We find that in all equilibria, high-score agents are ambitious, and low-score agents are practical. The analysis also uncovers three salient and surprising features of the symmetric equilibrium. One, the symmetric equilibrium necessarily involves agents randomizing between applications. Second, in this equilibrium, agents with scores on a continuous support sort into discrete classes, defined as intervals of priority scores, where they adopt exactly the same strategy. Third, the frictional market design is less efficient but more egalitarian than the benchmark design. Indeed, low-score agents may be better off with private information than with public information, because they benefit from a relative informational competitive advantage.

The value of this work is two-fold. It illustrates the role of information on priorities on matching markets, showing how the uncertainty interplays with other standard market frictions to distort the allocation. This results in a clear design recommendation: mechanism design, information design and priority design should be performed jointly in order to achieve the desired pattern of discrimination. It also makes a methodological contribution by displaying a novel and rich mode of strategic interactions arising within the agent side of a matching market (the application game), resulting in an equally novel equilibrium structure (the block structure).

We believe the lessons learned from this work may generalize to more sophisticated (idiosyncratic, hybrid) preferences and mechanism (dynamic, with deterministic and stochastic stages). Ultimately, our framework should be able to accommodate more numerous and interesting empirical applications: beyond social housing, any market where priority is defined by a cardinal point system (teacher allocation, college admissions in many countries).

Another interesting challenge in this research is to provide micro-foundations for the use of a sub-optimal mechanism. Ideally we would explicitly model the general design problem and show that once we consider a two-fold social objective (we care not only about efficiency but also about participation), the private information mechanism becomes optimal. Another foundation could come from political economy. In our applied example, the market frictions give a chance to middle class households to be allocated social housing. Using a frictional design could be a way for a greedy politician to earn these households' votes for reelection.

A Proofs

Proof of lemma 1.

- Action spaces $\forall\ i \in \{1,\dots,n\},\ A_i = \{A^1,\dots,A^m,N\}$ are finite. So by proposition 1 in Milgrom and Weber (1985) [18], payoffs are equicontinuous (R1).
- Types $\omega_i,\ i \in \{1,\dots,n\}$ are independent. So by proposition 3 in Milgrom and Weber (1985) [18], information is absolutely continuous (R2).

Finally, by theorem 1 in Milgrom and Weber (1985) [18] applied to the AG satisfying R1 and R2, there exists a BNE in the AG.

Proof of lemma 2.

- By definition, $\forall\ \omega \in [0,1],\ \mathbb{E}[u_i(N_i)|\omega] = 0$. This proves (i) and (ii) for $X_i = N_i$.
- For $X_i = A_i^j,\ j \in \{1,\dots,m\}$, we have:

$$\mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|\omega] = \underbrace{\prod_{i' \in \{1,\dots,n\}\setminus\{i\}} \left(\omega + \int_\omega^1 (1-p_{i'}^j(x))dx\right)}_{\mathbb{P}(S|A_i^j, p_{-i}([\omega,1]),\omega)} a^k - c$$

$\forall\ i' \in \{1,\dots,n\}\setminus\{i\},\ j \in \{1,\dots,m\},\ \omega \mapsto \int_\omega^1 (1-p_{i'}^j(x))dx$ is continuous.
So the probability of success $\omega \mapsto \mathbb{P}(S|A_i^j, p_{-i}([\omega,1]),\omega)$ and the whole expectation $\mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|\omega]$ are also continuous. This proves (i).
Set $\omega- < \omega+$.

$$\begin{aligned}
\omega- + \int_{\omega-}^1 (1-p_{i'}^j(x))dx &= \left(\omega+ + \int_{\omega+}^1 (1-p_{i'}^j(x))dx\right) + \omega- - \omega+ + \int_{\omega-}^{\omega+} (1-p_{i'}^j(x))dx \\
&= \left(\omega- + \int_{\omega-}^1 (1-p_{i'}^j(x))dx\right) + \int_{\omega-}^{\omega+} (-p_{i'}^j(x))dx \\
&< \omega+ + \int_{\omega+}^1 (1-p_{i'}^j(x))dx \\
&\implies \mathbb{P}(S|A_i^j, p_{-i}([\omega,1]),\omega-) < \mathbb{P}(S|A_i^j, p_{-i}([\omega,1]),\omega+) \\
&\implies \mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|\omega-] < \mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|\omega+]
\end{aligned}$$

This proves (ii).
In addition:

$$\lim_{\omega\to1} \mathbb{P}(S|A_i^j, p_{-i}([\omega,1]),\omega) = 1 \implies \lim_{\omega\to1} \mathbb{E}[u_i(A_i^j, p_{-i}([\omega,1]))|,\omega] = a^j\ \ (iii)$$

Proof of proposition 0.

For agent ranked i^{th} in priority, $i \in \{1,\dots,n\}$, denote $\mu(i) \in \{1,\dots,m,\emptyset\}$ her final allocation on the benchmark market.

(i) Induction statement: $H(i):\ \mu_i = i,\ i \in \{1,\dots,m\}$

- Initial step
  $\forall\ i\{1,\dots,n\}:\ \omega_1 > \omega_i \implies u_1(A_1^j,\sigma_{-1}) = a^j - c$
  $u^1(N_1,\sigma_{-1}) = 0$
  $\implies \forall\ \sigma_{-1},\ BR^1(\sigma_{-1}) = \{A_1^1\}$
  $\implies \sigma_1 = A_1^1,\ \mu_1 = 1\ H(1)$
- Inductive step
  Set $i \in \{1,\dots,m-1\}$ s.t. $H(1),\dots,H(i)$ true.
  $\forall\ j \in \{1,\dots,i\}:\ \omega_j > \omega_{i+1} \implies u_{i+1}(A_i^j,\sigma_{-(i+1)}) = -c$
  $\forall\ j \in \{i+1,\dots,n\}:\ \omega_{i+1} > \omega_j \implies u_{i+1}(A_i^j,\sigma_{-(i+1)}) = a^j - c$
  $u_i(N_i,\sigma_{-(i+1)}) = 0$
  $\implies BR_{i+1}(\sigma_{-(i+1)}) = \{A_{i+1}^{i+1}\}$
  $\implies \sigma_{i+1} = A_{i+1}^{i+1},\ \mu_{i+1} = i+1\ H(i+1)$

(ii) Set $m+1 \leq i \leq n$.
By (i): $\forall\ j \in \{1,\dots,m\},\ u_i(A_i^j,\sigma^{-i}) = -c$
$u_i(N_i,\sigma^{-i}) = 0$
$\implies BR_i(\sigma_{-i}) = \{N_i\}$
$\implies \sigma_i = N_i \implies \mu_i = \emptyset$

Proof of proposition 1.
We define the "interim action set" at score ω as the subset of actions that are played with positive probabilities at score ω: $A^j \in IAS(\omega)$ if $p^j(\omega) > 0$.
We state and prove a lemma characterizing interim action sets at any BNE of the AG (symmetric or asymmetric).

Lemma 4. *[Interim action sets at BNE]*
(i) At any BNE of the AG, $\sigma \in \square^{BNE}(\mathcal{G})$:
- *Robust profile:*
  - $\exists\ 1 = \omega^0 > \omega^1 > \omega^2 \geq 0$ *s.t.* : $\begin{cases} IAS((\omega^1,\omega^0)) = \{A^1\} \\ IAS((\omega^2,\omega^1)) = \{A^1,A^2\} \end{cases}$.
- *Potential profile:*
  - $\forall\ k \in \{3,m\}$, *if* $\exists\ \omega | p^k(\omega) < 0$, *then:*
    $\exists\ 1 = \omega^0 > \omega^1 > \dots > \omega^k \geq 0$ *s.t.:* $\forall\ j \in \{1,\dots,k\},\ IAS((\omega^j,\omega^{j-1})) = \{A^1,\dots,A^j\}$.
  - *If* $\exists\ \omega | p^{m+1}(\omega) < 0$, *then:*
    $\exists\ 1 = \omega^0 > \omega^1 > \dots > \omega^m > \omega^{m+1} = 0$ *s.t.* $\begin{cases} \forall\ j \in \{1,\dots,m\}, & IAS((\omega^j,\omega^{j-1})) = \{A^1,\dots,A^j\} \\ & \{N\} \subseteq IAS((0,\omega^m)) \end{cases}$

(ii) Moreover: ω^1 is the same in all multiple BNE of a given AG $\mathcal{G}$.

Proof of lemma 4. Set $\sigma \in \square^{BNE}(\mathcal{G})$.
- By lemma 2., (iii):

$$\lim_{\omega\to 1} \mathbb{E}[u_i|A_i^1,\sigma_{-i},\omega] = a^1 - c > \begin{cases} a^j - c = \lim_{\omega\to 1}\mathbb{E}[u_i|A_i^j,\sigma_{-i},\omega] \\ 0 = \lim_{\omega\to 1}\mathbb{E}[u_i|N,\sigma_{-i},\omega] \end{cases}$$

  And by lemma 2., (i) (continuity): $\exists\ \omega^1$ s.t. $IAS((\omega^1,1)) = \{A^1\}$.
- Suppose all players play A^1 at all scores: $IAS([0,1]) = \{A^1\}$.

$$\begin{aligned} \lim_{\omega\to 0}\mathbb{P}(S|A_i^1,\omega) = 0 &\implies \lim_{\omega\to 0}\mathbb{E}[u_i|A_i^1,\sigma_{-i},\omega] = -c < 0 \\ j \in \{2,\dots,m\}: \lim_{\omega\to 0}\mathbb{P}(S|A_i^j,\omega) = 1 &\implies \lim_{\omega\to 0}\mathbb{E}[u_i|A_i^j,\sigma_{-i},\omega] = a^j - c \\ \lim_{\omega\to 0}\mathbb{P}(S|N_i,\omega) = 0 &\implies \lim_{\omega\to 0}\mathbb{E}[u_i|N_i,\sigma_{-i},\omega] = 0 \end{aligned}$$

  Then, by lemma 2. again, (i) and (ii):
  - $\omega \mapsto \mathbb{E}[u_i|A_i^1,\sigma_{-i},\omega]$ is continuous and strictly increasing on $[0,1]$ from $-c$ to $a^1 - c$.
  - $\omega \mapsto \mathbb{E}[u_i|A_i^j,\sigma_{-i},\omega]$, $j \in \{2,\dots,m\}$ is constant on $[0,1]$ and equal to $a^j - c$.
  - $\omega \mapsto \mathbb{E}[u_i|N_i,\sigma_{-i},\omega]$, $j \in \{2,\dots,m\}$ is constant on $[0,1]$ and equal to 0.

  By the bijection theorem, $\exists\ \omega^1 \in [0,1]$ s.t.:

$$\mathbb{E}[u_i|A_i^1,\sigma_{-i},\omega^1] = \mathbb{E}[u_i|A_i^2,\sigma_{-i},\omega^1] = a^2 - c > \begin{cases} a^j - c = \mathbb{E}[u_i|A_i^j,\sigma_{-i},\omega], j \in \{3,\dots,m\} \\ 0 = \mathbb{E}[u_i|N,\sigma_{-i},\omega] \end{cases}$$

  And by lemma 2., (i) (continuity) again: $\exists\ 0 < \omega^2 < \omega^1$ s.t. $IAS((\omega^2,\omega^1)) \subseteq \{A^1,A^2\}$.
- Suppose no one plays A^k, $k \in \{1,\dots,2\}$ on (ω^2,ω^1). Set $k' \in \{1,2\}$, $\neq k$.
  Then, by lemma 2., (ii), again:
  - $\omega \mapsto \mathbb{E}[u_i|A_i^k,\sigma_{-i},\omega]$ is constant on (s^2,s^1) equals to $a^2 - c$.
  - $\omega \mapsto \mathbb{E}[u_i|A_i^{k'},\sigma_{-i},\omega]$ is strictly increasing on (s^2,s^1).

  So: $\mathbb{E}[u_i|A_i^k,\sigma_{-i},\omega] > \mathbb{E}[u_i|A_i^{k'},\sigma_{-i},\omega]$, and playing A^k is a profitable deviation. So $A^k \in IAS((\omega^1,\omega^2))$.
  Exchanging the the roles objects 1 and 2, we get: $\{A^1,A^2\} \subseteq IAS((\omega^1,\omega^2))$.
  In the end: $IAS((\omega^1,\omega^2)) = \{A^1,A^2\}$.

- The proof for the intervals (ω^k, ω^{k-1}), $j \in \{3, \dots, m\}$ below is similar. The inclusion $IAS(\omega^k, \omega^{k-1}) \subseteq \{A^1, \dots, A^k\}$ relies on the continuity and monotonicity of expected interim payoffs in lemma 2. enabling an application of the intermediate value theorem. The reverse inclusion comes from the indifference in ω^{k-1} plus the monotonicity of expected interim payoffs in lemma 2., giving a sharp characterization of the no profitable conditions.
- If ω^{m-1} exists and $\forall\ j \in \{1, \dots, m\}$ s.t. $\lim_{\omega \to 0} \mathbb{E}[u_i | A_i^j, \sigma_{-i}, \omega] < 0$, then by the intermediate value theorem again, $\exists\ 0 < \omega^m < \omega^{m-1}$ s.t. $\{N\} \subseteq SS((0, \omega^m)) \subseteq \{A^1, \dots, A^m, N\}$.

All the preceding proves (i) and (ii).

Notations: $s^k := \omega^k,\ k \in \{0, m+1\}$.
Lemma 4. shows that both actions A^1 and A^2 are played with positive probabilities on (s^2, s^1). At pure equilibrium, this implies that different players play different actions and the equilibrium profile is asymmetric.

Proof of theorem 1.

1. Interim action sets
   Interim action sets at symmetric equilibrium are given by lemma 4..
   Notations: $t^k := \omega^k,\ k \in \{0, m+1\}$.

2. Constant probabilities
   Let us now locate on the interval (t^k, t^{k-1}) and prove that the probability functions $\omega \mapsto p^j(\omega),\ j \in \{1, m\}$ are constant on each interval (t^k, t^{k-1}). Due to the complexity of notations, we write down the explicit proof of the probabilities being constant at the inductive step for $k = 2$ and give the way to go for the lower classes.
   - Set $k = 2$. Let us locate on the (t^2, t^1) interval.
     The strong indifference principle applied at a score $\omega^\star \in (t^2, t^1)$ delivers the following differential equation:

$$
\begin{aligned}
(E^1_{[\![1,2]\!]}) &: \mathbb{E}[u_i(A_i^1(\omega), \sigma_{-i}([\omega^\star, 1]))|\omega^\star] = \mathbb{E}[u_i(A_i^2(\omega), \sigma_{-i}([\omega^\star, 1]))|\omega^\star] \\
&\iff \left(1 - \int_{t^1}^1 f(\omega)d\omega - \int_{\omega^\star}^{t^1} p^1(\omega) f(\omega) d\omega\right)^{n-1} a^1 - c = \left(1 - \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega\right)^{n-1} a^2 - c \\
&\iff \left(1 - F(1) + F(t^1) - F(t^1) + F(\omega^\star) + \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega\right)^{n-1} a^1 \\
&\quad = \left(1 - \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega)\right)^{n-1} a^2 \\
&\iff F(\omega^\star) + \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega = \left(\frac{a^2}{a^1}\right)^{\frac{1}{n-1}} \left(1 - \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega\right) \\
&\iff (1 + F(t^1)) \int_{\omega^\star}^{t^1} p^2(\omega) f(\omega) d\omega = F(t^1) - F(\omega^\star)
\end{aligned}
$$

     Set: G^2, a primitive of $p^2 f$. Then:

$$
(E^1_{[\![1,2]\!]}) \iff G^2(t^1) - G^2(\omega^\star) = \frac{F(t^1) - F(\omega^\star)}{1 + F(t^1)}
$$

     Deriving on both sides, we get a necessary condition on the probability functions:

$$
\begin{aligned}
(E^1_{[\![1,2]\!]}) &\implies p^2(\omega^\star) f(\omega^\star) = \frac{-f(\omega^\star)}{1 + F(t^1)} \\
&\implies \begin{cases} p^2(\omega^\star) = \dfrac{1}{1 + F(t^1)} \\ p^1(\omega) = \dfrac{F(t^1)}{1 + F(t^1)} \end{cases}
\end{aligned}
$$

We now need to check that those constant probability functions indeed verify equation $(E^1_{[\![1,2]\!]})$:

$$(1+F(t^1))\int_{\omega^\star}^{t^1}\frac{1}{1+F(t^1)}f(\omega)d\omega = F(t^1)-F(\omega^\star)$$
$$\iff \frac{(1+F(t^1))}{(1+F(t^1))}(F(t^1)-F(\omega^\star)) = F(t^1)-F(\omega^\star)\ \checkmark$$

- Set $k\in\{2,\dots,m-1\}$, and suppose $H(1),\dots,H(k)$ hold. Let us locate on the interval (t^{k+1},t^k). The strong indifference principle applied at a score $\omega^\star\in(t^{k+1},t^k)$ delivers a system of k differential equations with $k+1$ unknowns. We denote those equations $(E^l_{[\![1,k+1]\!]})$, $j\in\{1,\dots,k\}$. Each of them is given by:

$$(E^j_{[\![1,k+1]\!]}): \mathbb{E}[u_i(A_i^j(\omega),\sigma_{-i}([\omega^\star,1]))|\omega^\star] = \mathbb{E}[u_i(A_i^{k+1}(\omega),\sigma_{-i}([\omega^\star,1]))|\omega^\star]$$
$$\iff \Big(1-\int_{\omega^\star}^{t^k}p^j(\omega)f(\omega)d\omega-\sum_{l=j}^{t^k}\int_{t^l}^{t^{l-1}}p^j(\omega)f(\omega)d\omega\Big)^{n-1}a^j-c$$
$$=\Big(1-\int_{\omega^\star}^{t^k}p^{k+1}(\omega)f(\omega)d\omega\Big)^{n-1}a^{k+1}-c$$

If we replace $p^1(\omega^\star)$ by $1-\sum_{j=2}^{k+1}p^k(\omega^\star)$, we end up with only k unknowns, hence a Cramer system. We can use the substitution method, to get in the end a relation between (for instance) only $\int_{\omega^\star}^{t^k}p^{k+1}(\omega)f(\omega)d\omega$ and $p^k_{[\![1,k]\!]}\int_{t^k}^{t^{k-1}}f(\omega)d\omega$. Posing primitives and deriving the whole gives constant probabilities $p^j_{[\![1,k+1]\!]}$.

3. Probability levels:
Set $k\in\{2,\dots,m-1\}$, and further exploit the differential equations:

$$(E^j_{[\![1,k+1]\!]}): \mathbb{E}[u_i(A_i^j(\omega),\sigma_{-i}([\omega^\star,1]))|\omega^\star] = \mathbb{E}[u_i(A_i^{k+1}(\omega),\sigma_{-i}([\omega^\star,1]))|\omega^\star]$$
$$\iff \Big(1-\int_{\omega^\star}^{t^k}p^j(\omega)f(\omega)d\omega-\sum_{l=j}^{t^k}\int_{t^l}^{t^{l-1}}p^j(\omega)f(\omega)d\omega\Big)^{n-1}a^j-c$$
$$=\Big(1-\int_{\omega^\star}^{t^k}p^{k+1}(\omega)f(\omega)d\omega\Big)^{n-1}a^{k+1}-c$$
$$\iff \Big(1-\int_{\omega^\star}^{t^k}p^j(\omega)f(\omega)d\omega-\sum_{l=j}^{t^k}\int_{t^l}^{t^{l-1}}p^j(\omega)f(\omega)d\omega\Big)$$
$$=\Big(\frac{a^{k+1}}{a^j}\Big)^{\frac{1}{n-1}}\Big(1-\int_{\omega^\star}^{t^k}p^{k+1}(\omega)f(\omega)d\omega\Big)$$
$$\iff \Big(1-p^j_{[\![1,k+1]\!]}(F(\omega^\star)-F(t^k))-\sum_{l=j}^{t^k}p^j_{[\![1,l]\!]}((F(t^l)-F(t^{l-1}))\Big)$$
$$=\Big(\frac{a^{k+1}}{a^j}\Big)^{\frac{1}{n-1}}\Big(1-p^{k+1}_{[\![1,k+1]\!]}(F(\omega^\star)-F(t^k))\Big)$$

Deriving on both sides, we get:

$$(E^j_{[\![1,k+1]\!]}) \implies -p^j_{[\![1,k+1]\!]}f(\omega^\star) = -\Big(\frac{a^{k+1}}{a^j}\Big)^{\frac{1}{n-1}}p^{k+1}_{[\![1,k+1]\!]}f(\omega^\star)$$
$$\implies p^j_{[\![1,k+1]\!]} = \Big(\frac{a^{k+1}}{a^j}\Big)^{\frac{1}{n-1}}p^{k+1}_{[\![1,k+1]\!]}$$

This is a recursive formula for the probability levels within class (t^{k+1},t^k). To find the explicit formulas, we use the fact that the $k+1$ probability levels sum up to one:

$$\begin{aligned}\sum_{j=1}^{k+1} p^j_{[\![1,k+1]\!]} = 1 &\implies p^{k+1}_{[\![1,k+1]\!]} \sum_{l=1}^{k+1} \Big(\frac{a^{k+1}}{a^l}\Big)^{\frac{1}{n-1}} = 1 \\ &\implies p^{k+1}_{[\![1,k+1]\!]} = \Big((a^{k+1})^{\frac{1}{n-1}} \sum_{l=1}^{k+1} (a^l)^{\frac{-1}{n-1}}\Big)^{-1} \\ &\implies p^{j}_{[\![1,k+1]\!]} = \Big((a^{j})^{\frac{1}{n-1}} \sum_{l=1}^{k+1} (a^l)^{\frac{-1}{n-1}}\Big)^{-1} \\ &\implies p^{j}_{[\![1,k+1]\!]} = \Big(\sum_{l=1}^{k+1} \Big(\frac{a^j}{a^l}\Big)^{\frac{1}{n-1}}\Big)^{-1}\end{aligned}$$

Finally, on the bottom interval $(0,t^m)$, it cannot be that players apply to $j \in \{1,\dots,m\}$. Otherwise, by lemma 2. (ii), $\omega \mapsto \mathbb{E}[u|A^j,\sigma,\omega]$ would be increasing so strictly negative at some score $\in (0,t^m)$, hence a profitable deviation to action N. So: p^{m+1} is constant equal to 1 on $(0,t^m)$.

4. Thresholds:
   The remaining task is to characterize the thresholds $t^k,\ k \in \{2,\dots,m\}$.
   - $t^k,\ k \in \{2,\dots,m-1\}$ By definition:

$$t^k := \inf\{\omega^\star \in [0,1] |\ \forall\ \omega > \omega^\star,\ \min_{l\in\{1,\dots,k\}} \mathbb{E}[u_i(A_i^l(\omega),\sigma_{-i}([\omega,1]))|\omega] \geq \mathbb{E}[u_i(A_i^{k+1}(\omega),\sigma_{-i}([\omega,1]))|\omega]\}$$

   By the strong indifference principle, we have that all $\mathbb{E}[u_i(A_i^l(\omega),\sigma_{-i}([\omega,1]))|\omega],\ l \in \{1,\dots,k\}$ are equal on (t^k,t^{k-1}). So:

$$\begin{aligned}&t^k = \inf\{\omega^\star \in [0,1] |\ \forall\ \omega > \omega^\star,\ \mathbb{E}[u_i(A_i^k(\omega),\sigma_{-i}([\omega,1]))|\omega] \geq \mathbb{E}[u_i(A_i^{k+1}(\omega),\sigma_{-i}([\omega,1]))|\omega]\} \\ &\implies \mathbb{E}[u_i(A_i^k t^k),\sigma_{-i}([t^k,1]))|t^k] = \mathbb{E}[u_i(A_i^{k+1}(t^{k+1}),\sigma_{-i}([t^{k+1},1]))|t^{k+1}] \\ &\implies \Big(1-\int_{t^k}^{t^{k-1}} p^k(\omega)f(\omega)d\omega\Big)^{n-1} a^k - c = a^{k+1} - c \\ &\implies \big(1-p^k_{[\![1,k]\!]}(F(t^{k-1})-F(t^k)\big)^{n-1} a^k - c = a^{k+1} - c \\ &\implies p^k_{[\![1,k]\!]}(F(t^{k-1})-F(t^k)) = 1-\Big(\frac{a^{k+1}}{a^k}\Big)^{\frac{1}{n-1}} \\ &\implies F(t^k) = F(t^{k-1}) + \frac{1}{p^k_{[\![1,k]\!]}}\Big(-1+\Big(\frac{a^{k+1}}{a^k}\Big)^{\frac{1}{n-1}}\Big)\end{aligned}$$

   Plugging in the formula for the probability levels, we get:

$$F(t^k) = F(t^{k-1}) + (a^k)^{\frac{1}{n-1}} \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}\Big(-1+\Big(\frac{a^{k+1}}{a^k}\Big)^{\frac{1}{n-1}}\Big)$$
$$F(t^k) = F(t^{k-1}) + ((a^{k+1})^{\frac{1}{n-1}} - (a^k)^{\frac{1}{n-1}}) \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}$$

   This is a recursive formula characterising the thresholds. The explicit formula is therefore:

$$F(t^k) = F(t^0) + \sum_{j=1}^{k} ((a^{j+1})^{\frac{1}{n-1}} - (a^j)^{\frac{1}{n-1}}) \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}$$

   The second term is close to be a telescopic sum. The second part is $\sum_{j=1}^k \alpha_j (a^j)^{\frac{1}{n-1}}$ where we compute the terms below:
   - $j=1$: $\alpha_1(a^1)^{\frac{1}{n-1}} = -(a^1)^{\frac{1}{n-1}}(a^1)^{\frac{-1}{n-1}} = -1$
   - $2 \leq j \leq k$:
     $$\alpha_j(a^j)^{\frac{1}{n-1}} = \Big(\sum_{l=1}^{j-1}(a^l)^{\frac{1}{n-1}} - \sum_{l=1}^{j}(a^l)^{\frac{-1}{n-1}}\Big)(a^j)^{\frac{-1}{n-1}} = -(a^j)^{\frac{1}{n-1}}(a^j)^{\frac{-1}{n-1}} = -1$$
   - $j=k+1$: $\alpha_{k+1}(a^{k+1})^{\frac{1}{n-1}} = \sum_{l=1}^{k}(a^l)^{\frac{-1}{n-1}}(a^{k+1})^{\frac{1}{n-1}} = \sum_{l=1}^{k}\Big(\frac{a^{k+1}}{a^l}\Big)^{\frac{1}{n-1}}$

In the end, we get:

$$F(t^k) = 1 - k + \sum_{l=1}^{k} \Big(\frac{a^{k+1}}{a^l}\Big)^{\frac{1}{n-1}}$$

- $k = m$

$$\begin{aligned}
&\mathbb{E}[u_i(A_i^m(t^m), \sigma_{-i}([t^m,1]))|t^m)] = \mathbb{E}[u_i(N(t^m), \sigma_{-i}([t^m,1]))|t^m)] \\
\iff &\big(1 - p^m_{[\![1,m]\!]}(F(t^m) - F(t^{m-1}))\big)^{n-1} a^m - c = 0 \\
\iff &F(t^m) = F(t^{m-1}) + \frac{1}{p^m_{[\![1,m]\!]}}\Big(-1 + (\frac{c}{a^m})^{\frac{1}{n-1}}\Big) \\
\iff &F(t^m) = 2 - m + \sum_{l=1}^{m-1} (\frac{a^m}{a^l})^{\frac{1}{n-1}} + \sum_{l=1}^{m} \Big(\frac{a^m}{a^l}\Big)^{\frac{1}{n-1}} \Big(-1 + (\frac{c}{a^m})^{\frac{1}{n-1}}\Big) \\
\iff &F(t^m) = 1 - m + \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}}
\end{aligned}$$

5. Number of classes

$$\begin{aligned}
k_0(p^\star) = k \in \{1, \dots, m\} &\iff F(t^k) \le 0 < F(t^{k-1}) \\
&\iff 1 - k + \sum_{l=1}^{k} \Big(\frac{a^{k+1}}{a^l}\Big)^{\frac{1}{n-1}} \le 0 < 2 - k + \sum_{l=1}^{k-1} \Big(\frac{a^k}{a^l}\Big)^{\frac{1}{n-1}} \\
&\iff 1 + \sum_{l=1}^{k} \Big(\frac{a^{k+1}}{a^l}\Big)^{\frac{1}{n-1}} \le k < 2 + \sum_{l=1}^{k-1} \Big(\frac{a^k}{a^l}\Big)^{\frac{1}{n-1}} \\
k_0(p^\star) = m + 1 &\iff F(t^m) > 0 \\
&\iff 2 - m + \sum_{l=1}^{m} \Big(\frac{c}{a^l}\Big)^{\frac{1}{n-1}} > 0 \\
&\iff 2 + \sum_{l=1}^{m} \Big(\frac{c}{a^l}\Big)^{\frac{1}{n-1}} > m
\end{aligned}$$

Proof of corollary 1.

(i) Probability level variations

- Within classe
  By the block structure, we immediately have that for $\omega,\ \omega' \in [t^k, t^{k-1}]:\ p^j(\omega) = p^j_{[\![1,k]\!]} = p^j(\omega')$.
- Between classes
  From theorem 1., probability levels write as:

$$p^j_{[\![1,k]\!]} = \Big((a^j)^{\frac{1}{n-1}} \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}\Big)^{-1}$$

  $k \mapsto \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}$ is a sum with positive terms, hence increasing in k. Going to the inverse, we find that $p^j_{[\![1,k]\!]}$ is decreasing in k.

(ii) First Order Stochastic Dominance
Denote: $\sigma(\omega)$ the distribution with support $\{A^1, \dots, A^m\}$ and probabilities: $\mathbb{P}(\sigma(\omega) = A^j) = p^j(\omega),\ j \in \{1, \dots, m\}$.
Using the formulas for probability levels in theorem1., we find for $\omega \in (t^k, t^{k-1})$:

$$\begin{aligned}
\sum_{l=1}^{j} p^l(\omega) = \sum_{l=1}^{j} p^l_{[\![1,k]\!]} &= \begin{cases} \sum_{l=1}^{j} \Big((a^j)^{\frac{1}{n-1}} \sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}\Big)^{-1} & \text{if } j < k \\ 1 & \text{if } j \ge k \end{cases} \\
&= \begin{cases} \dfrac{\sum_{l=1}^{j} (a^l)^{\frac{-1}{n-1}}}{\sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}} & \text{if } j < k \\ 1 & \text{if } j \ge k \end{cases}
\end{aligned}$$

Set $0 \leq \omega' < \omega \leq 1$.
We seek to demonstrate $\forall\ j \in \{1,\ldots,m\}$:

$$(\star)_j : \sum_{l=1}^{j} p^l(\omega') \leq \sum_{l=1}^{j} p^l(\omega)$$

- If ω', ω belong to the same class C^k, then because of the block structure:

$$\sum_{l=1}^{j} p^l(\omega') = \sum_{l=1}^{j} p^l(\omega)$$

  So $(\star)_j,\ j \in \{1,\ldots,m\}$ trivially holds.
- If ω', ω belong to different classes, $\omega' \in C^{k'},\ \omega \in C^k,\ k < k'$, then there are two subcases:
  - If $k' = m+1$ then $\forall\ j \in \{1,\ldots,m\},\ \sum_{l=1}^{j} p^l(\omega') = 0$ and $(\star)_j$ is trivially verified.
  - If $k' \leq m$ then:
    For $j \leq k$, then: $\sum_{l=1}^{j} p^l(\omega) = 1$ and $(\star)_j$ is trivially verified.
    For $j < k < k'$, then:

$$\frac{\sum_{l=1}^{j} p^l(\omega)}{\sum_{l=1}^{j} p^l(\omega')} = \frac{\sum_{l=1}^{k'} (a^l)^{\frac{-1}{n-1}}}{\sum_{l=1}^{k} (a^l)^{\frac{-1}{n-1}}} > 1 \implies (\star)_j$$

(iii) From the proof of theorem 1., we have the following recursive formula:

$$p^j_{[\![1,k]\!]} = \left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}} p^k_{[\![1,k]\!]}$$

And $a^j \mapsto \left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}}$ is decreasing in j, equivalently increasing in j. So, $p^j_{[\![1,k]\!]}$ is increasing in j.

(iv) ex ante probability of applying to object j:

$$p^j := \sum_{k=j}^{m+1} (F(t^{k-1}) - F(t^k)) p^j_{[\![1,k]\!]}$$

For $1 \leq k \leq m-1$, from theorem 1. we have:

$$\begin{aligned}
(F(t^{k-1}) - F(t^k)) p^j_{[\![1,k]\!]} &= \frac{1}{p^k_{[\![1,k]\!]}} \left(1 - \left(\frac{a^{k+1}}{a^k}\right)^{\frac{1}{n-1}}\right) p^j_{[\![1,k]\!]} \\
&= \sum_{l=1}^{k} \left(\frac{a^k}{a^l}\right)^{\frac{1}{n-1}} \frac{1}{\sum_{l=1}^{k} \left(\frac{a^j}{a^l}\right)^{\frac{1}{n-1}}} \left(1 - \left(\frac{a^{k+1}}{a^k}\right)^{\frac{1}{n-1}}\right) \\
&= \left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}} \left(1 - \left(\frac{a^{k+1}}{a^k}\right)^{\frac{1}{n-1}}\right) \\
&= \frac{(a^k)^{\frac{1}{n-1}} - (a^{k+1})^{\frac{1}{n-1}}}{(a^j)^{\frac{1}{n-1}}} \quad (\star)
\end{aligned}$$

Summing up, we recognize a telescopic sum and we get:

$$p^j = \frac{(a^j)^{\frac{1}{n-1}} - (a^{m+1})^{\frac{1}{n-1}}}{(a^j)^{\frac{1}{n-1}}}$$

$$p^j = 1 - \left(\frac{c}{a^j}\right)^{\frac{1}{n-1}}$$

Proof of corollary 2.

(i) All agents participate iff there are at most m classes. By theorem 1. (i), this happens iff:

$$m \geq 1 + \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}}$$

(ii) By proposition 0., at the Nash equilibrium of the Application Game with perfect information, the expected participation is $\frac{m}{n}$.
By theorem 1., at the symmetric BNE of the Application Game with imperfect information, expected participation is:

$$1 - F(t^m) = m - \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}} = \sum_{l=1}^{m} 1 - \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}} = \sum_{l=1}^{m} p^l$$

Participation is therefore higher at BNE iff:

$$m - \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}} \geq \frac{m}{n}$$
$$1 - \frac{1}{n} \geq \frac{1}{m} \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}}$$
$$m \geq \frac{n}{n-1} \sum_{l=1}^{m} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}}$$

Proof of proposition 2.

(B) By proposition 0., the welfare on the benchmark market is given by:

$$W^B := \mathbb{E}[u(\sigma^*)] = \frac{1}{n} \sum_{k=1}^{m} (a^k - c)$$

Since $\forall\ k \in \{1, \ldots, m\},\ a^k - c > 0$, this is the maximum welfare attainable in the Application Game.

(F) We use the formula of the interim welfare, $t^k < w < t^{k-1}$ (see section §5.2):

$$W^F(\omega) = \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \cdot \Big(\big(\frac{a^k}{a^j}\big)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^{n-1} a^j - c$$

The ex ante welfare aggregates all interim welfare ($k = m$) taking into acount the distribution of scores:

$$\begin{aligned} W^F &= \int_0^1 W^F(\omega) f(\omega) d\omega \\ &= \sum_{k=1}^{m} \Big[\int_{t^k}^{t^{k-1}} \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \cdot \Big(\big(\frac{a^k}{a^j}\big)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^{n-1} a^j f(\omega) d\omega - c(F(t^{k-1}) - F(t^k)) \Big] \end{aligned}$$

We denote:

$$\begin{aligned} I_k &:= \int_{t^k}^{t^{k-1}} \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \cdot \Big(\big(\frac{a^k}{a^j}\big)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^{n-1} a^j f(\omega) d\omega \\ &= \sum_{j=1}^{k} p^j_{[\![1,k]\!]} a^j \cdot \int_{t^k}^{t^{k-1}} \Big(\big(\frac{a^k}{a^j}\big)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^{n-1} f(\omega) d\omega \end{aligned}$$

We denote:

$$
\begin{aligned}
L_{kj} &:= \int_{t^k}^{t^{k-1}} \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^{n-1} f(\omega) d\omega \\
&= \Big[\frac{1}{np^j_{[\![1,k]\!]}} \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} \Big)^n \Big]_{t^{k-1}}^{t^k} \\
&= \frac{1}{np^j_{[\![1,k]\!]}} \Big[(\frac{a^k}{a^j})^{\frac{n}{n-1}} - \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(t^k)) p^j_{[\![1,k]\!]} \Big)^n \Big] \\
&\overset{(\star)}{=} \frac{1}{np^j_{[\![1,k]\!]}} \Big[(\frac{a^k}{a^j})^{\frac{n}{n-1}} - \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - \frac{(a^k)^{\frac{1}{n-1}} - (a^{k+1})^{\frac{1}{n-1}}}{(a^j)^{\frac{1}{n-1}}} \Big)^n \Big] \\
&= \frac{1}{np^j_{[\![1,k]\!]}} \Big[(\frac{a^k}{a^j})^{\frac{n}{n-1}} - \Big((\frac{a^{k+1}}{a^j})^{\frac{1}{n-1}} \Big)^n \Big] \\
&= \frac{1}{np^j_{[\![1,k]\!]}} \Big[\frac{(a^k)^{\frac{n}{n-1}} - (a^{k+1})^{\frac{n}{n-1}}}{(a^j)^{\frac{n}{n-1}}} \Big]
\end{aligned}
$$

Substituting in I_k, we get:

$$
\begin{aligned}
I_k &= \sum_{j=1}^{k} p^j_{[\![1,k]\!]} a^j . \frac{1}{np^j_{[\![1,k]\!]}} \Big[\frac{(a^k)^{\frac{n}{n-1}} - (a^{k+1})^{\frac{n}{n-1}}}{(a^j)^{\frac{n}{n-1}}} \Big] \\
&= \frac{1}{n} ((a^k)^{\frac{n}{n-1}} - (a^{k+1})^{\frac{n}{n-1}}) \sum_{j=1}^{k} (a^j)^{\frac{-1}{n-1}}
\end{aligned}
$$

Substituting in W^F, we get:

$$
W^F = \frac{1}{n} \Big(\sum_{k=1}^{m} ((a^k)^{\frac{n}{n-1}} - (a^{k+1})^{\frac{n}{n-1}}) \sum_{j=1}^{k} (a^j)^{\frac{-1}{n-1}} \Big) - c \sum_{k=1}^{m} (F(t^{k-1}) - F(t^k))
$$

The second term is a telescopic sum and the first term is close tot be a telescopic sum.

- $k = 1$: $(a^1)^{\frac{n}{n-1}} . (a^1)^{\frac{-1}{n-1}} = a^1$
- $2 \leq k \leq m$: $(a^k)^{\frac{n}{n-1}} \big(\sum_{j=1}^{k} (a^j)^{\frac{-1}{n-1}} - \sum_{j=1}^{k-1} (a^j)^{\frac{-1}{n-1}} \big) = (a^k)^{\frac{n}{n-1}} (a^k)^{\frac{-1}{n-1}} = a^k$
- $k = m+1$: $-(a^{m+1})^{\frac{n}{n-1}} \sum_{j=1}^{m} (a^j)^{\frac{-1}{n-1}}$

In total, we get:

$$
\begin{aligned}
W^F &= \frac{1}{n} \big(\sum_{k=1}^{m} a^k - (a^{m+1})^{\frac{n}{n-1}} \sum_{j=1}^{m} (a^j)^{\frac{-1}{n-1}} \big) - c(1 - F(t^m)) \\
&= \frac{1}{n} \big(\sum_{k=1}^{m} a^k - c^{\frac{n}{n-1}} \sum_{j=1}^{m} (a^j)^{\frac{-1}{n-1}} \big) - c \Big(m - \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big) \\
&= \frac{1}{n} \big(\sum_{k=1}^{m} a^k \big) - \frac{1}{n} c \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} - c \Big(m - \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big) \\
&= \frac{1}{n} \big(\sum_{k=1}^{m} a^k \big) - c \Big(m - \frac{n}{n-1} \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big)
\end{aligned}
$$

The welfare gap is given by:

$$
\begin{aligned}
W^B - W^F &= \frac{1}{n} \sum_{j=1}^{m} (a^j - c) - \Big[\frac{1}{n} \sum_{k=1}^{m} a^k + c \Big(m - \frac{n-1}{n} \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big) \Big] \\
&= c \Big(-\frac{m}{n} + m - \frac{n-1}{n} \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big) \\
&= c \frac{n-1}{n} \Big(m - \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Big)
\end{aligned}
$$

$$
\forall l \in \{1, \ldots, m\}, c < a^l \Rightarrow (\frac{c}{a^l})^{\frac{1}{n-1}} < 1 \Rightarrow m > \sum_{l=1}^{m} (\frac{c}{a^l})^{\frac{1}{n-1}} \Rightarrow W^B > W^F
$$

Proof of lemma 3.

(B) By proposition 0., on at the Nash equilibrium of the benchmark market, and agent receives utility $(a^i - c)$ iff he is ranked i^{th} in priority. For an agent with score ω, this happens with probability:

$$\mathbb{P}(\omega \text{ ranked } i) = (1 - F(\omega))^{i-1} F(\omega)^{n-i} \binom{n-1}{i-1}$$

In total, we have:

$$W^B(\omega) = \sum_{i=1}^{m} \binom{n-1}{i-1} (1 - F(\omega))^{i-1} F(\omega)^{n-i} (a^i - c)$$

F cdf hence continuous $\Rightarrow W^B$ continuous.
By definition (or by computation, using the binomial theorem):

$$\sum_{i=1}^{n} \binom{n-1}{i-1} (1 - F(\omega))^{i-1} F(\omega)^{n-i} = 1$$

When ω increases, $\mathbb{P}(\omega$ ranked $i)$ increases (decreases) for small (large) i - associated to high (low) utilities $(a^i - c)$. So $W^B(\omega)$ strictly increasing with ω.

(F) By theorem 1., for $t^k < \omega < t^k - 1$:

$$W^F(\omega) := \mathbb{E}[u(p^\star)|\omega] = \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \mathbb{P}(S|A^j, \omega) a^j - c$$

$$\mathbb{P}(S|A^j, \omega) = \Big(1 - \sum_{l=j}^{k-1} (F(t^{l-1}) - F(t^l)) p^j_{[\![1,l]\!]} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]}\Big)^{n-1}$$

From the proof of corollary 1., we know:

$$(F(t^{l-1}) - F(t^l)) p^j_{[\![1,l]\!]} = \frac{(a^l)^{\frac{1}{n-1}} - (a^{l+1})^{\frac{1}{n-1}}}{(a^j)^{\frac{1}{n-1}}} \quad (\star)$$

Summing up, we recognize a telescopic sum and we get:

$$\sum_{l=j}^{k-1} (F(t^{l-1}) - F(t^l)) p^j_{[\![1,l]\!]} = \frac{(a^j)^{\frac{1}{n-1}} - (a^k)^{\frac{1}{n-1}}}{(a^j)^{\frac{1}{n-1}}} = 1 - (\frac{a^k}{a^j})^{\frac{1}{n-1}}$$

Substituting, we get:

$$\mathbb{P}(S|A^j, \omega) = \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]}\Big)^{n-1)}$$

$$W^F(\omega) := \mathbb{E}[u(p^*)|\omega] = \sum_{j=1}^{k} p^j_{[\![1,k]\!]} a^j \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]}\Big)^{n-1} - c$$

By theorem 1. again, for $0 \le \omega \le t^m$, we trivially get $W^F(\omega) = 0$ (constant).
F cdf hence continuous $\Rightarrow W^F$ continuous.
For the monotonicity, we derive:

$$\frac{\partial \mathbb{P}(S|A^j, \omega)}{\partial \omega} = f'(\omega)(F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]} (n-1) \Big((\frac{a^k}{a^j})^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega)) p^j_{[\![1,k]\!]}\Big)^{n-2} > 0$$

This immediately proves the monotonicity within class.
When combined with corollary 1. (ii), it also proves the monotonicity across classes.

Proof of proposition 3.

We define the interim welfare gap: $g(\omega) = W^B(\omega) - W^F(\omega)$.
By theorem 1.:

$$m > 1 + \sum_{l=1}^{k} \left(\frac{c}{a^l}\right)^{\frac{1}{n-1}} \iff W^F(0) > 0 \iff g(0) > 0$$

By proposition 2. (continuity), we get existence of the threshold ω'.

Proof of proposition 4.

The proof is similar to the proof of theorem 1..

- On class 1 to m, the endogenous cost simplifies in the differential equations, and we get the same system than with exogenous cost.
- In the equation for the threshold t^m differs, the cost does not simplify:

$$\begin{aligned}
&\mathbb{E}[u_i(A_i^m(t^m), \sigma_{-i}([t^m,1]))|t^m)] = \mathbb{E}[u_i(N(t^m), \sigma_{-i}([t^m,1]))|t^m)] \\
\iff &\left(1 - p^m_{[\![1,m]\!]}(F(t^m) - F(t^{m-1}))\right)^{n-1} a^m - c(t^m) = 0 \\
\iff &F(t^m) = 1 - m + \sum_{l=1}^{m} \left(\frac{c(t^m)}{a^l}\right)^{\frac{1}{n-1}}
\end{aligned}$$

The difference in $F(t^m)$ in the endogenous cost model vs exogenous cost model is given by:

$$(c(t^m) - c) \sum_{l=1}^{m} \left(\frac{1}{a^l}\right)^{\frac{1}{n-1}} > 0 \iff c(t^m) > c$$

Proof of corollary 3.

The interim expected payoff with endogenous cost is given by $(,\omega \in [t^k, t^{k-1}],\ k \in \{1, \ldots, m\})$:

$$\begin{aligned}
W_e^F(\omega) &:= \sum_{j=1}^{k} p^j_{[\![1,k]\!]} \cdot \left(\left(\frac{a^k}{a^j}\right)^{\frac{1}{n-1}} - (F(t^{k-1}) - F(\omega))\, p^j_{[\![1,k]\!]}\right)^{n-1} - c(\omega) \\
\frac{\partial W_e^F(\omega)}{\partial \omega} &= \frac{\partial W^F(\omega)}{\partial \omega} - \frac{dc(\omega)}{d\omega}
\end{aligned}$$

$c()$ decreasing with ω ($\frac{dc(\omega)}{d\omega} < 0$) implies $\frac{\partial W_e^F(\omega)}{\partial \omega} > \frac{\partial W^F(\omega)}{\partial \omega}$.

Proof of proposition 5.

The interim expected payoffs at score 1 write:

$$\begin{aligned}
\mathbb{E}[u_i(A_i^1(1))|1] &= a^1 - c \\
\mathbb{E}[u_i(A_i^2(1))|1] &= a^2 - c \\
\mathbb{E}[u_i(B_i(1))|1] &= a^1 - 2c \\
\mathbb{E}[u_i(N_i(1))|1] &= 0
\end{aligned}$$

So $\sigma_i^\star(1) = A^1$. By continuity, $\exists \omega^1 < 1$ s.t. $\sigma_i^\star((\omega^1, 1]) = A^1$.
The interim payoffs at lower scores write:

$$\begin{aligned}
\mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}[\omega,1])|\omega] &= \omega^2 a^1 - c \\
\mathbb{E}[u_i(A_i^2(\omega), \sigma^\star_{-i}[\omega,1])|\omega] &= a^2 - c \\
\mathbb{E}[u_i(B_i(\omega), \sigma^\star_{-i}[\omega,1])|\omega] &= \omega^2 a^1 + (1-\omega^2) a^2 - 2c \\
\mathbb{E}[u_i(N_i(\omega), \sigma^\star_{-i}[\omega,1])|\omega] &= 0
\end{aligned}$$

We solve indifference equations:

$$\mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}[\omega,1])|\omega] = \mathbb{E}[u_i(A_i^2(\omega), \sigma^\star_{-i}[\omega,1])|\omega] \iff \omega = \sqrt{\frac{a^2}{a^1}}$$
$$\mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}[\omega,1])|\omega] = \mathbb{E}[u_i(B_i(\omega), \sigma^\star_{-i}[\omega,1])|\omega] \iff \omega = \sqrt{1-\frac{c}{a^1}}$$
$$\sqrt{\frac{a^2}{a^1}} < \sqrt{1-\frac{c}{a^1}} \iff \frac{a^2}{a^1} + \frac{c}{a^2} > 1$$

We get two cases:

- $\frac{a^2}{a^1} + \frac{c}{a^2} > 1$: Below a threshold $t^1 = \sqrt{\frac{a^2}{a^1}}$, players switch to playing A^1 and A^2 with indifference.
  Just below t^1 and similarly to the case with truncation, the agents plays A^2 with probability $p^2_{[\![1,2]\!]} = \frac{1}{1+t^1}$.
  At a score $\omega < t^1$, interim payoffs write:
  $$\mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = \mathbb{E}[u_i(A_i^2(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = (1-(t^1-\omega)p^2_{[\![1,2]\!]})^2 a^2 - c = \big(1-(1-t^1)-(t^1-\omega)p^1_{[\![1,2]\!]}\big)^2 a^1 - c$$
  $$\mathbb{E}[u_i(B(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = \big(1-(1-t^1)-(t^1-\omega)p^1_{[\![1,2]\!]}\big)^2 a^1 + \big(1-(t^1-\omega)p^2_{[\![1,2]\!]}\big)a^2 - 2c$$
  $$\Delta(\omega) := \mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] - \mathbb{E}[u_i(B(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = c - \big(1-(t^1-\omega)p^2_{[\![1,2]\!]}\big)^2 a^2$$
  $$\frac{\partial\Delta}{\partial\omega} = -2p^2_{[\![1,2]\!]}\big(1-(t^1-\omega)p^2_{[\![1,2]\!]}\big)^2 a^2 < 0$$
  So Δ is decreasing until t^1. And by definition of this case: $\Delta(t^1) = 0$. So on the left of t^1, $\Delta(\omega) > 0$. The agent does not switch to B. Just as in the model with truncation, he randomizes between A^1 and A^2 potentially until a threshold t^2 where he starts playing N. Below t^2 all interim payoffs stay constant, so the agent keeps on playing N until score 0.
- $\frac{a^2}{a^1} + \frac{c}{a^2} < 1$: Below a threshold $r^1 = \sqrt{1-\frac{c}{a^2}}$, players switch to playing B.
  At a score $\omega < r^1$, interim payoffs write:
  $$\mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = \omega^2 a^1 - c$$
  $$\mathbb{E}[u_i(B(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] = \omega^2 a^1 + \big((1-r^1)^2 + 2\omega(1-\omega)\big)a^2 - 2c$$
  $$\Delta(\omega) := \mathbb{E}[u_i(B(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] - \mathbb{E}[u_i(A_i^1(\omega), \sigma^\star_{-i}([\omega,1]))|\omega] := \big((1-r^1)^2 + 2\omega(1-\omega)\big)a^2 - c$$
  $$\frac{\partial\Delta}{\partial\omega} = 2(1-2\omega)a^2 > 0 \iff \omega < \frac{1}{2}$$
  So Δ is decreasing on the left neighborhood of r^1. And by definition of this case: $\Delta(r^1) = 0$. So on the left neighborhood of r^1, $\Delta(\omega) > 0$, and the agent does not immediately switch to another action.

Proof of proposition 6.

The interim expected payoffs at score 1 write:

$$\mathbb{E}[u_i(\oplus(1)_i)|1] = v - c$$
$$\mathbb{E}[u_i(\ominus(1)_i)|1] = u - c$$

So $\sigma^\star(1) = \oplus$. By continuity: $\exists t^1 < 1$ s.t. $\sigma^\star((t^1,1]) = \oplus$
The interim expected payoffs at score $\omega < 1$ write:

$$\mathbb{E}[u_i(\oplus_i(\omega), \sigma^\star_{-i})|\omega] = (1-(1-\omega)\theta)v - c$$
$$\mathbb{E}[u_i(\ominus_i(\omega), \sigma^\star_{-i})|\omega] = (1-(1-\omega)(1-\theta))u - c$$

We characterize the threshold point t^1 where the agent start being indifferent between the two actions:

$$\mathbb{E}[u_i(\oplus_i(t^1), \sigma^\star_{-i})|t^1] = \mathbb{E}[u_i(\ominus_i(t^1), \sigma^\star_{-i})|t^1] \iff (1-(1-t^1)\theta)v = (1-(1-t^1)(1-\theta))$$
$$\iff t^1 = \frac{\theta u - (1-\theta)v}{\theta v + (1-\theta)u}$$

We find $t^1 > 0 \iff \theta > \frac{v}{v+u}$, and $\frac{v}{v+u} > \frac{1}{2}$ so the condition is non trivial.
The interim expected payoffs at score $\omega < t^1$ write:

$$\mathbb{E}[u_i(\oplus_i(\omega), \sigma^\star_{-i})|\omega] = [\omega + ((1-t^1) + (t^1-\omega)p(\oplus))(1-\theta) + (t^1-\omega)(1-p(\oplus))\theta]\,v - c$$
$$\mathbb{E}[u_i(\oplus_i(\omega), \sigma^\star_{-i})|\omega] = [\omega + ((1-t^1) + (t^1-\omega)p(\oplus))\theta + (t^1-\omega)(1-p(\oplus))(1-\theta)]\,u - c$$

The probability level $p(\oplus)$ making the agent indifferent between the two actions is characterized by:

$$\mathbb{E}[u_i(\oplus_i(\omega), \sigma^\star_{-i})|\omega] = \mathbb{E}[u_i(\ominus_i(\omega), \sigma^\star_{-i})|\omega]$$
$$\iff [\omega + ((1-t^1) + (t^1-\omega)p(\oplus))(1-\theta) + (t^1-\omega)(1-p(\oplus))\theta]\,v = [\omega + ((1-t^1) + (t^1-\omega)p(\oplus))\theta + (t^1-\omega)(1-p(\oplus))(1-\theta)]\,u$$

We derive this equation and get:

$$[1 - p(\oplus)(1-\theta) - (1-p)\theta]\,v = [1 - p\theta - (1-p)(1-\theta)]\,u$$
$$\iff p(\oplus) = \frac{\theta u - (1-\theta)v}{(2\theta-1)(u+v)}$$

We check that this level indeed verifies the indifference equation.
The interim payoff at scores below t^1 strictly decreases and may hit the zero bound at some lower score t^2, where the agent with lower score would decide not to apply.

***** Proofs of toy examples (§4.2) *****

Proof of example $n = 3 > m = 2, F \sim \mathscr{U}$ for (asymmetric) pure BNE

- Top class $(s^1, 1)$:
  By lemma 4., $\forall\ i \in \{1, \dots, 3\}, \sigma_i((s^1, 1)) = A^1$.
- Middle class (s^2, s^1):
  By lemma 4.:
  $\exists\ i \in \{1,2,3\}, s_i((s^2, s^1)) = A^1$ (set i=1).
  $\exists\ i' \in \{1,2,3\}, s_i((s^2, s^1)) = A^2$ (set i'=2).
  Let us characterize the action of player 3 on $[s^2, s^1]$. The interim payoffs of player 3 at $\omega' < s^1$ write:

  $$\mathbb{E}[u_3(A^1_3(\omega'), p([\omega', s^1])|\omega'] = \omega' s^1 a^1 - c$$
  $$\mathbb{E}[u_3(A^2_3(\omega'), p([\omega', s^1])|\omega'] = (1 - (s^1 - \omega'))a^2 - c$$

  The difference between the two is:

  $$\Delta(\omega') = [\omega' s^1 a^1 - c] - [(1-(s^1-\omega'))a^2 - c] = \omega' s^1 a^1 - (1 - s^1 + \omega')a^2$$

  We differentiate with respect to ω':

  $$\frac{\partial \Delta(\omega')}{\partial \omega'} = s^1 a^1 - a^2 = \sqrt{a^1 a^2} - a^2 = \sqrt{a^2}(\sqrt{a^1} - \sqrt{a^2}) > 0$$

  So Δ is strictly increasing. We know, by indifference at s^1: $\Delta(s^1) = 0$. So: $\Delta(\omega') < 0$.
  Conclusion: $s_3([s^2, s^1]) = A^2$.

- Bottom interval $(0, s^2)$:
  By lemma 4.: $\exists\ i \in \{1,2,3\}$ s.t. $s^i([0, s^2]) = N$.
  By lemma 2. (ii): $\omega \mapsto \mathbb{E}[u_1(A^1(\omega), s([\omega,])|\omega]$ is constant..
  So $s_1([0, s^2]) = A^1$ and $i \neq 1$.
  Fix $i = 2$.
  By lemma 2. (i) and (ii): $\begin{cases} \omega \mapsto \mathbb{E}[u_2(N(\omega), s([\omega,])|\omega] \text{ is constant.} \\ \omega \mapsto \mathbb{E}[u_3(A^2(\omega), s([\omega,1])|\omega] = \mathbb{E}[u_3(N(\omega), s([\omega,1])|\omega] \text{ is constant.} \\ \omega \mapsto \mathbb{E}[u_1(A^1(\omega), s([\omega,1])|\omega] \text{ is constant.} \end{cases}$

So: $s_2([0, s^2]) = N$ or A^2.

Proof of example $n = 3 > m = 2, F \sim \mathscr{U}$ for symmetric (interior) BNE
Included in theorem 1..

B Supplements

B.1 Pure (asymmetric) Bayes-Nash equilibrium - Partial characterization

The next theorem only partially characterizes the pure BNE of the AG:

Theorem 2. *[Pure (asymmetric) BNE]*
A pure strategy BNE of the AG:

(i) exists and is unique up to strategies on the $(0, s^m)$ *interval, and payoff-unique.*

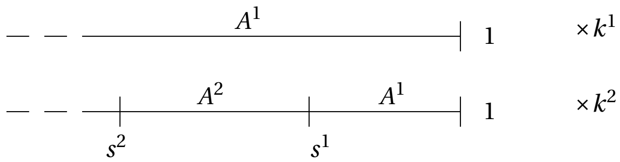
(ii) exhibits finite number of intervals of scores where the interim action sets are constant.

The proof is in two steps. First, we use lemma 4 characterizing the interim action sets. Second, for each interval of score with constant interim action set, we characterize the number of agents playing each action in the interim action set. For example, on the interval $[s^2, s^1]$, we determine the pair(s) of two integers (k^1, k^2), $k^1 + k^2 = n$, where k^1 (k^2) of agents playing A^1 (A^2). We find that the no profitable deviation inequalities between payoffs always defines a (unique) pair (k^1, k^2). We proceed similarly at lower scores.

The theorem still allows many different patterns within the intervals where the interim action set is constant. Whenever we introduce $n \geq 4$ agents, the equilibrium patterns depends finely on the parameters of the AG, hence a low robustness.[30] We illustrate this lack of robustness below with an example:

$n = 4, F \sim \mathcal{U}$

- Case 1: $a^1 > 8a^2 + 7c \rightarrow (k^1, k^2) = (1, 3)$
- Case 2: $a^1 < 8a^2 + 7c \rightarrow (k^1, k^2) = (2, 2)$



In general, the pure BNE can support quite odd strategy profiles, where some strategies exhibit no sorting (an agent plays higher value objects at lower scores), or sorting with jumps (an agent plays high value objects at high scores, low value objects at intermediate scores but never plays the intermediary value objects). In these profiles, each strategy is virtually unique and highly sophisticated. The profiles are "very assymmetric". This questions the ability of players to coordinate on these equilibria.

B.2 Symmetric (interior) equilibrium - Comparative statics

In this section, we describe how a change in the parameters of the AG affects the symmetric BNE of the AG.

Values and cost

The coming proposition emphasizes that the symmetric equilibrium is invariant to a rescaling of all object values and the application cost.

Proposition 7. *[Invariance to rescaling]*
In an Application Game, if we multiply all object values and the cost by a given constant, the symmetric (interior) Bayes-Nash equilibrium remains unchanged.

Proposition 7 implies that any non-trivial comparative static analysis must first keep the cost fixed as object values fluctuate, and second, normalize the object values while varying the cost.

The relative position of values affects the equilibrium structure in an intuitive way. If objects are highly homogeneous in values, agents almost perfectly randomize between available objects. There is close to a single class, the lowest one, plus the no application class. Coordination is horizontal. When, on the contrary, the values are heterogeneous, people sort by levels of scores, the strategy is close to being pure. Coordination is vertical.

[30] In this respect, the robustness of the pure equilibrium in section §4.2 was a special feature of the toy example.

When the cost is high, agents even with intermediate scores resort to the safe no application action, and agents with the highest score coordinate. When the cost is low, incentives to play the no application action disappear, and incentives to coordinate on different objects are reduced.

We provide below a graphical representation of the discussion in the toy example.

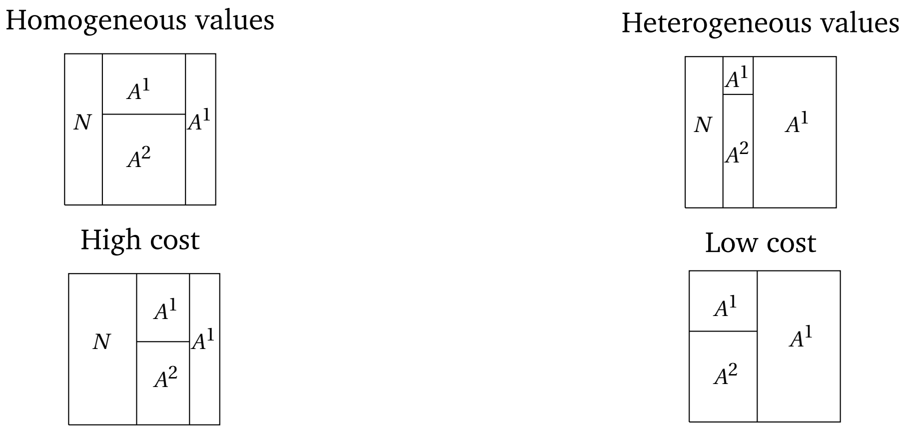


FIGURE 17: Comparative statics: Symmetric (interior) BNE for various values and cost - $m = 2$

Market balance

An increase in the number of agents n increases all thresholds, decreases class sizes. The effect is especially strong on bottom classes. It increases (decreases) probability levels for low (high) value objects. In net effect, introducing more agents decreases the ex ante probability with which any action is played, except for the no application action, which is played more frequently. The rationale is that more numerous agents generate competition via an increase in the probability of crowding, hence in the occurrence of failure. This pushes agents to be more cautious: high score agents mix with lower value objects, and low score agents more often decide not to apply.

Due to the recursive structure of the AG, the introduction of an additional object has a very clean effect on the equilibrium structure, described in the coming proposition.

Proposition 8. *[Effect of additional object]*
In an Application Game, the addition of an object with a given value $a^{k_0+1} < a^{new} < a^{k_0}$:

- *Only affects equilibrium thresholds t^k and levels $p^j_{[\![1,k]\!]}$ on the adjacent higher class and on lower classes ($k \in \{k_0, \dots, m\}$).*
- *Does not affect the equilibrium application probabilities for all application actions A^j, $j \in \{1, \dots, m\}$ that were already available before the addition.*

In other words, the addition of an object does not affect individual behavior at levels of scores where agents were all applying to higher-value objects. It does affect individual behavior at levels of scores where some were applying to lower value objects. The collective behavior remains unchanged in the sense that each lower value object is played as often as before the addition: only the identities of the applicants are modified, not the mass. Playing the new object only happens at the detriment of the no application strategy.

Priority score distribution

Another key property of the equilibrium is that the whole effect of the distribution is captured in the thresholds hence in the class sizes. The next proposition formalizes this remark.

Proposition 9. *[Effect of priority score distribution]*
In an Application Game, the priority score distribution:

- *Does not affect equilibrium levels.*
- *Only affects the equilibrium thresholds, in a way that keeps the mass of each class fixed.*

In summary, we expect a narrow (wide) class at score levels featuring a high (low) concentration of agents. Whether narrow with many agents or wide with few agents, an equilibrium class always features the same mass. In particular, if we change the priority score distribution to a mean-preserving spread distribution, we will get that extreme (middle) classes become narrower (wider). In expectation, the number of agents playing each possible mixture will remain unchanged.