Ratings Design and Barriers to Entry

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Abstract

I study the impact of consumer reviews on the incentives for firms to participate in the market through the lens of ratings design. Firms produce goods of heterogeneous, unknown quality that is gradually revealed via consumer reviews, and face both entry and exit decisions. A platform combines past reviews to construct firm-specific ratings that help guide consumer search. When the platform integrates all reviews into ratings – full transparency – consumers form queues at the highest-rated firms. This demand profile induces an S-shaped continuation value for firms as a function of ratings. Whereas firms thus prefer more feedback when struggling and less feedback when established, equilibrium induces precisely the reverse allocation. The main insight of the paper is that suppressing the reviews of highly-rated firms makes the task of climbing the ratings ladder less arduous, thus stimulating participation and consumer welfare. This insight is robust to extensions that allow for price setting, costly effort and heterogeneous consumer preferences.

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1 Introduction

In this paper, I ask: can consumer reviews create barriers to entry for new firms? To answer this question, I develop a model in which firms of unknown quality make entry and exit decisions. Quality is gradually revealed through consumer reviews. Reviews are controlled by a platform that uses them to construct firm-specific ratings. These ratings shape consumer demand. The main insight of the paper is that suppressing reviews for well-established incumbents can stimulate incentives for new firms to enter and remain active, ultimately leading to gains in consumer welfare. Studying the interaction between ratings design – and information design more broadly – and industry dynamics is novel and constitutes the main contribution of the paper.

Product ratings systems – platforms that aggregate user-generated feedback to help inform consumer choice – are ubiquitous, playing a significant role in shaping choices and transforming the fortunes of all involved. Such platforms provide an indispensable source of information, reducing search frictions and informational asymmetries and thereby allowing consumers and producers to engage in profitable trade.[1] Indeed, well-established firms and products often have many hundreds of reviews to their name, affording consumers unprecedented precision when making purchases. But while this stockpile of information might serve incumbent firms to great effect, it might work against a new entrant who unavoidably starts from scratch.

Consider a recently-opened restaurant. Initially, their meal quality is unknown, but to make matters worse, a nearby restaurant has a reasonable rating on Yelp! with hundreds of reviews. Given these options, consumers will more likely choose the latter, simply through judicious application of Bayes' Rule – the newcomer is assumed to be of average quality, having not had the chance to build a credible reputation.[2] In light of this exacting consumer behavior, the new entrant could shut down prematurely, perhaps after a few poor reviews. Indeed, they might not enter the market in the first place, given the severity of

[1] Examples include Amazon, Yelp!, Google Reviews, Zomato, TripAdvisor and RateMDs to list just a few. According to recent surveys, over 90% of consumers now consult online reviews before making purchase decisions. Displaying reviews can increase purchasing rates by 270% - see http://spiegel.medill.northwestern.edu/_pdf/Spiegel_Online%20Review_eBook_Jun2017_FINAL.pdf. Various empirical studies document the importance of consumer reviews in determining firm revenue (e.g. Andersen and Magruder (2012), Lewis and Zervas (2016), Luca (2016)). Fradkin (2018) and Farronato and Fradkin (2018) provide evidence that such platforms reduce trading frictions and significantly improve consumer welfare. Tadelis and Zettelmeyer (2015) propose a theory and supporting evidence on how the information provided by such platforms helps trading counter-parties to match efficiently, increasing overall surplus. Finally, see Tadelis (2016) for an excellent survey of both supporting theories and evidence.

[2] The problem might be exacerbated in reality by heuristic updating procedures that give undue weight to a greater mass of reviews regardless of perceived quality (Powell1 et al. (2017)).

the initial conditions and sizable start-up costs. The following quote from the website of Zomato, a restaurant review platform, echoes this narrative: *"The penalty from a bad review could have been a death sentence, especially for a new place…as a low rating may prevent new customers from visiting the restaurant."*[3] It could be argued that this is not a problem, given that the incumbent had a reasonable rating and so is likely to be of good quality. But this argument fails to consider that the demoralized entrant might provide a superior quality product without having had the chance to prove their worth. These characteristics could apply to a variety of markets, such as medical services, hotels, labor markets and traditional product markets; in the context of digital music platforms, a recent study by Aguiar and Waldfogel (2018) shows how such forces can ultimately lead to severely reduced consumer welfare.

These issues relate to the well-documented "cold start"phenomenon, a self-reinforcing link between a lack of sampling and a lack of information regarding product quality (Che and Hörner (2018), Kremer et al. (2014)). These studies treat the range of available products as exogenously given, abstracting from producer participation constraints. My analysis captures a novel equilibrium feedback channel through which cold-starting endogenously determines the distribution of product quality, which in turn determines the relative demand and feedback rates for new products.

The model comprises consumers, firms and a ratings platform. Firms produce output of heterogeneous quality. Output is stochastic, and depends on the firm's underlying type, which is either high or low. Each firm's type is unknown to all market participants, and is gradually revealed through a *rating* publicly provided by the platform. Firms pay a fixed cost to enter, subsequently incurring a constant operating cost, and are subject to a service capacity constraint. Once active, firms decide if and when to irreversibly exit the market. A fixed measure of short-lived consumers choose between all available firms by using the rating provided to engage in frictionless, directed search, subject to firms' capacity constraints and random rationing. Each consumer truthfully reports their experience to the platform, who uses a firm's history of such reviews to form its rating.

I begin by fully characterizing the unique stationary equilibrium of the economy under a regime of *full transparency*, in which all consumer reviews received by the platform are incorporated into a firm's rating (Theorem 1). Equilibrium under this regime features congestion by consumers (Lemma 3) who trade off expected quality against the probability of service and thus visit the highest-rated firms and queue for

[3] see https://www.zomato.com/blog/helping-new-restaurateurs-find-their-feet. In a similar vein, a recent article describes a single bad review on TripAdvisor as *"…the marketing PR equivalent of a drive-by shooting"*, with the modern consumer labeled as *"…a veritable tyrant, with the power to make or break lives."*

their services. The exacting nature of consumer choice gives rise to a non-convexity in the firm's problem: its continuation value from remaining operational is S-shaped in its rating (Lemma 4). Importantly, from a firm's perspective, consumer feedback when poorly-rated comes with *upside gain*; a standard option value effect emerges whereby the gains from a positive review are potentially large, but the losses are ameliorated by the possibility of exit. Conversely, for highly-rated firms feedback entails *downside loss* as the firm experiences diminishing marginal returns from a higher rating due to capacity constraints. As such, struggling firms want rapid feedback, whereas successful firms want minimal feedback. Crucially, under full transparency, *precisely the reverse profile* obtains, as successful firms attract more customers and thus more reviews. This informational misallocation depresses firms' incentives, resulting in reduced entry and increased exit rates.

From this benchmark, a number of testable implications are derived. The model shares many predictions with well-known models of firm dynamics. For instance, it predicts that average quality increases with age, that exit hazard rates are hump-shaped in age, and that the full invariant distribution of firms over ratings is right-skewed due to selection effects. (see Jovanovic (1982), Luttmer (2007)) However, it produces several predictions that are novel: due to rapid feedback for highly-rated firms, the ratings distribution has a fatter right tail than left (Corollary 1). Both the sell-out and feedback rates are increasing functions of a firm's rating, and the fraction of consumers unable to secure a purchase is an increasing, strictly convex function of the firm's rating (Lemma 3).

With these insights in hand, I turn to the prudent design of ratings systems. In equilibrium, the difficulties faced by firms inexorably feed into consumer welfare. Thus, while I model the platform's objective as being consumer welfare, its choice of policy should also account for firms' incentives. I assume that the platform is limited in its choice of instrument: it must respect consumers' desires to sample whichever firms they please, but is able to control the inclusion of consumer reviews into firms' ratings. Thus, rather than simply report the entire history of reviews for each firm, the platform can commit to a filtering of this history, in the spirit of Hörner and Lambert (2018). Ratings design in this context then has the ability to shape the market profile by providing firms with incentives to participate; to my knowledge, my paper is the first to study the interplay of information design and industry dynamics.

The central insight from this normative analysis is that simple *upper censorship* policies can improve consumer welfare, by excluding reviews from the ratings process of highly-rated firms. In the benchmark model, this feature is stark: within a broad class of policies, the optimal system involves maximal feedback for low-rated firms, and minimal feedback for high-rated firms (Theorem 2). The intuition is simple yet compelling. In order to maximize their welfare, the platform wants consumers to buy from only the highest rated firms. This harms young firms, as they enjoy profits only if they achieve this high standard. Without further intervention, entry rates would be depressed and exit would be rapid, ultimately adversely affecting consumers themselves. Since the platform cannot offer direct transfers, nor can they send consumers to firms they are unwilling to visit, they must shore up incentives for firms through careful design of the ratings system. Throwing away reviews for well established, highly rated firms not only prevents their rating from soaring further, but also prevents their rating from sliding. For struggling firms, this makes climbing the ratings ladder both easier and more rewarding once conquered, thus providing the necessary encouragement. To test the robustness of this result, I explore several extensions, the most important of which is allowing firms to post prices. Prices provide an important intensive margin through which firms may calibrate their terms of trade. For instance, new firms might price at a loss in order to attract consumers and feedback, while established firms might increase prices in order to soak up the extra demand generated by higher ratings. I show that both effects emerge under full transparency. However, I also show that yet again there exist simple censorship policies that improve upon full transparency.

The paper proceeds as follows. Section 2 introduces the model and solves for equilibrium outcomes under a benchmark regime of *full transparency*. Section 3 turns to ratings design. Section 4 considers various extensions of the baseline model, including price setting, effort and heterogeneous consumer preferences. Section 5 provides a detailed review of the relevant literature and places the contribution of the paper within it.

2 Benchmark Model - Full Transparency

Time is continuous and doubly infinite. The economy consists of positive measures of firms and consumers, as well as a platform. As I will be working with stationary equilibria, time subscripts are dropped, with t henceforth denoting the age of a firm. I begin by abstracting from the design question by assuming that the platform simply includes all consumer feedback into each firm's rating.

Firms – A large, infinitely elastic supply of firms can potentially participate in market production. Each firm can be one of two types, $\theta \in \{0, 1\}$. Types are fixed throughout the life of a firm and are initially hidden, with a new entrant being of type θ_h with probability p_0. While active, a firm of type θ

and age t is associated with stochastic process $(X_t)_{t\geqslant 0}$ that evolves according to

$$dX_t = \lambda_t \theta dt + \sqrt{\lambda_t}\sigma dZ_t \tag{1}$$

where $(Z_t)_{t\geqslant 0}$ is a Wiener process independent of θ, $\sigma \in (0,\infty)$, and the process λ_t is non-zero, with $\frac{1}{\lambda_t}$ is locally integrable and square integrable. (see Engelbert and Schmidt (1991) for details) Formally, let $(\Omega, \Sigma, \mathbb{P})$ be a probability space rich enough to admit Z, and let $\mathbb{E}$ denote the unconditional expectation operator under $\mathbb{P}$. Let $\mathbb{F}^x \triangleq \{\mathcal{F}_t^x\}_{t\geqslant 0}$ denote the natural filtration generated by $(X_t)_{t\geqslant 0}$. Finally, $\mathbb{E}_t^x$ denote the conditional expectation under $\mathbb{P}$ with respect to $\mathcal{F}_t^x$, so that $p_t \triangleq \mathbb{E}_t^x(\theta)$ denotes the instantaneous probability that the firm is of high type given the information contained in X_t. Henceforth I refer to p_t as a firm's *rating*.

An infinitely elastic supply of firms is available at any instant to enter the market. Firms pay an entry cost of $K > 0$. Once active, firms pay a flow cost $c > 0$ and can choose to irreversibly exit the market at any time. The rate at which firms serve consumers is denoted by π_t, which I further assume to be the firm's revenue.[4] Crucially, π_t is endogenously determined by consumer choice, but taken as given by a firm.Firms are subject to a service capacity constraint $\bar{\lambda}$. Finally, firms discount at rate $\rho > 0$ and face a constant hazard-rate $\delta > 0$ of exogenous attrition. I make the following minimal assumptions on firms' costs – were either of these assumptions violated, no firm would ever enter the market:[5]

Assumption 1. $\bar{\lambda} \in (c, \infty)$

Assumption 2. $K < \bar{V} \triangleq \frac{\bar{\lambda}-c}{\rho+\delta}$

Ratings – Throughout Section 2, I take $\lambda_t = \pi(p_t) + \epsilon$, for some $\epsilon > 0$. This specification admits a simple, established micro-foundation.[6] $(X_t)_{t\geqslant 0}$ can be interpreted as *cumulative review process*, where $\pi(p_t)$ denotes the rate at which reviews are left through endogenous consumer choice, and ϵ represents background learning generated by un-modelled consumers that do not use the platform to guide their search, visiting firms at random and leaving feedback nonetheless. (see Che and Hörner (2018) for a similar approach. Implicitly, I assume that background consumers generate zero revenue. This is without loss - one could simply incorporate this gain into c.) That π_t is assumed to be purely a function of the

[4] As such, prices are fixed at unity. In Section 4.1, prices are fully endogenized.

[5] The expression $\bar{V}$ is the present-discounted value of selling out forever, and thus forms an upper bound on incumbent firms' continuation values.

[6] See Bergemann and Välimäki (1997), Moscarini and Smith (2001) and Bolton and Harris (1999) for identical approaches.

firm's rating will be shown to be without loss, as p is the only payoff-relevant state variable for consumers. (see Lemma 3)

The exit decision of firms takes the form of a standard optimal stopping problem, for which the rating p_t forms a natural state variable. Applying Theorem 9.1 in Lipster and Shryaev (1977), noting that the capacity constraint ensures that λ is locally bounded and so Itô's Lemma can be applied.

Lemma 1. *Ratings evolve according to the SDE:*

$$dp_t = \frac{\sqrt{\lambda_t}}{\sigma} p_t(1 - p_t)d\bar{Z}_t \tag{2}$$

where $d\bar{Z}_t$ is a standard $\mathbb{F}^x$-adapted Wiener process.

Thus, λ_t can be thought of as measuring the rate at which reviews pass into the stock X_t and so controls the speed at which a firm's rating evolves.

Profits – The present value to a firm with rating p is

$$v(p) = \sup_{\tau^x} \mathbb{E}^x \left[\int_0^{\tau^x} e^{-(\rho+\delta)t}(\pi(p_t) - c)dt | p_0 = p \right] \tag{3}$$

where the supremum is taken over all $\mathbb{F}^x$-measurable stopping times.

Standard verification theorems exist for this setting (Rüschendorf and Urusov (2008)), yielding that the firm's exit strategy takes the form of a *rating threshold* $\underline{p} \in (0, 1]$, combined with a second-order ODE that expresses the firm's continuation value $V(p)$ from remaining operational, given a current rating of p.

Lemma 2 (Incumbent's Problem)**.** *Suppose there exists $\omega < p_0$ such that $\pi(p) < c$ for all $p \in [0, \omega]$. Let the pair $\{u(.), \underline{p}\}$ for $u \in \mathcal{C}^1([0, 1])$ and $\underline{p} \in [0, 1]$ denote the variational inequality:*

$$\begin{aligned} \mathcal{A}\{u\} &= 0 \\ u(\underline{p}) &= 0 \\ u'(\underline{p}) &= 0 \\ u(p) &\geqslant 0 \quad \forall p \in [\underline{p}, 1] \\ u(p) &= 0 \quad \forall p \in [0, \underline{p}], \end{aligned} \tag{4}$$

where

$$\mathcal{A}\{u\} = \pi(p) - c + \Sigma(p)u''(p) - (\rho+\delta)u(p), \quad \Sigma(p) = \frac{1}{2\sigma^2}p^2(1-p)^2\lambda(p) \tag{5}$$

Suppose $\{V(.), \underline{p}\}$ solves the problem (4), (5) with respect to the process $(p_t)_{t\geqslant 0}$ and that $\tau = \inf_s\{s \geqslant 0 : p_s \leqslant \underline{p}\}$. Then the function $V(.)$ coincides with the value $v(.)$ in (3), τ attains the supremum in (3), and $\{V(.), \underline{p}\}$ are unique.

Finally, an infinitely elastic supply of firms leads to a free entry condition on the present value to an entrant:

$$V(p_0) = K, \tag{6}$$

that in turn pins down the equilibrium rate of entry η.

Ratings Distribution – The combination of evolving ratings and a continuously churning positive mass of firms gives rise to an equilibrium firm distribution f. Since ratings are described by a diffusion process, they are Markovian and strongly recurrent. As such, the invariant distribution F^∞ for the ratings process is also ergodic, admitting a density almost everywhere with support $[\underline{p}, 1]$. Denote this density by f^∞. Since the mass of firms endogenously determined, and thus generically not equal to unity, f is a re-scaling of f^∞ determined by aggregate flow conditions detailed below. We abuse terminology and refer to f itself as the *ratings distribution*. The law of motion for this distribution follows the Fokker-Planck forward equation, with stationarity imposed and subject to various boundary conditions.

Proposition 1. *Let $f(p)$ denote the ratings distribution. Then $f(p) = 0$ for all $p \in [0, \underline{p})$. For almost all $p \in [\underline{p}, 1]$, f satisfies the Fokker-Planck forward equation:*

$$\frac{\partial^2}{\partial p^2}\Sigma(p)f(p) - \delta f(p) = 0 \tag{7}$$

subject to the following conditions:

(F_1) $f(\underline{p}) = 0$

(F_2) $\Sigma(p_0)[f'_-(p_0) - f'_+(p_0)] = \eta$

(F_3) $\Sigma(1)f(1) = 0$

(F_4) $\frac{\partial}{\partial p}\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, p_0) \cup (p_0, 1])$

(F_5) $\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, 1])$

(F_6) $\delta \int_{\underline{p}}^1 f(p)dp + \Sigma(\underline{p})f'_+(\underline{p}) = \eta$

While technical, many of these conditions provide economic insight into the dynamical system when properly interpreted. Condition F_1 is the classical "attainable boundary"condition, stating that firms spend no time at the exit threshold $\underline{p}$. Condition F_2 states that the rate at which incumbent firms move away from the initial rating p_0 must equal the rate of inflow by new entrants. Condition F_6 is an aggregate balance equation, analogues of which can be found in models of labor search and matching, e.g. Moscarini (2005). It states that outflows due to both attrition and voluntary exit must equal inflows.

Consumers – A unit measure of consumers use the platform to find firms. If a consumer purchases from a firm with type θ, they receive a payoff that is normally distributed with mean θ and variance σ^2. Given the available choices $f(p)$, consumers perform frictionless directed search subject to random rationing. (see Guerrieri and Shimer (2013), Lester (2011)) That is, if a consumer chooses to direct their search to firms with rating p, they are served at a rate:

$$\Theta(p) = \min \left\{ \frac{\bar{\lambda} f(p)}{g(p)}, 1 \right\} \tag{8}$$

where $g(p)$ is the density of consumers also searching within this submarket. Combining this matching technology with the assumption of frictionless search and risk-neutrality yields the well-known indifference condition on the equilibrium value J achieved by consumers:

$$J = \max_{p \in [\underline{p}, 1]} \left[p\Theta(p) \right], \tag{9}$$

subject to the market clearing condition:

$$\int_{\underline{p}}^{1} g(p) dp = 1 \tag{10}$$

2.1 Discussion

The benchmark model was constructed to balance the objectives of allowing the key economic forces to speak clearly while incorporating sufficient richness to render the analysis robust. However, given its novelty, a brief discussion of its main features is warranted.

The assumption that prices are fixed and exogenous is clearly restrictive. In certain settings, such as the market for medical services, prices are fixed or highly inflexible from the viewpoint of consumers. Allowing new entrants to price below marginal cost to attract consumers and thus feedback might provide the informational advantage they desire.The most natural model of price formation in such a setting would be competitive search price posting. I fully solve this extended model, showing how upper censorship policies yet again dominate full transparency; see Section 4.1 for details.

Modeling consumers as performing frictionless directed search is both tractable and realistic, capturing several intuitive features of platform search; a consumer opens Yelp!, and is greeted by a list of all available restaurants, along with a rating for each one. They then decide which one to go to, trading off quality against congestion - better restaurants typically involve longer service times. Such patterns echo recent empirical findings - Horton (2018) and Fradkin (2018) document pervasive congestion externalities in the context of an online platform. The idea that consumers trade off congestion against quality seems natural, and indeed has empirical support in the context of restaurants (Andersen and Magruder (2012)). Finally, directed search easily allows price setting to be built into the model.

The assumption of binary underlying quality whilst unrealistic affords several advantages. Firstly, it is highly tractable – beliefs regarding θ are summarized by a scalar. Were a model with a richer quality distribution (e.g. if $\theta \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$) used, not only would the exit problem of the firm become intractable (the exit threshold would transform into a higher-dimensional surface), but the ratings distribution would be itself be multi-dimensional. Indeed, in these richer settings, it is possible for old firms to be of low quality, which would appear to exacerbate the entry problem at the heart of the paper. Secondly, the binary assumption captures the crucial feature of interest, namely that in equilibrium, entrants are on average closer to the margin of exit than the average incumbent, i.e. $p_0 < \mathbb{E}(f)$, due to ex-post selection. Simply put, it only requires a few bad reviews for a young firm to give up, whereas an established incumbent can accommodate a run of bad luck with greater ease.

2.2 Stationary Equilibrium

I study stationary equilibria of the above model:

Definition 1. A stationary equilibrium is a collection of functions $\{V, f, g\}$ defined on $[0,1]$ and scalars $\{J, \underline{p}, \eta\}$ such that equations (4), (6), (7), (9), (10) and the relevant conditions listed in Lemma 2 and Proposition 1 all hold.

I start by solving for consumers' equilibrium choices $g(p)$. This is a potentially complicated problem, as there exists a non-trivial fixed-point relationship between g and the distribution f – note that equation

(7) governing f depends on $\lambda(p)$, which in turn depends on $g(p)$ and thus on f through equation (9). However, the solution turns out to take a simple *threshold strategy* form:

Lemma 3. *In equilibrium,*

$$g(p) = \begin{cases} \frac{\bar{\lambda} p f(p)}{p^*} & \text{if } p \geqslant p^* \\ 0 & \text{if } p < p^* \end{cases} \tag{11}$$

for some $p^* \in (\underline{p}, 1]$.

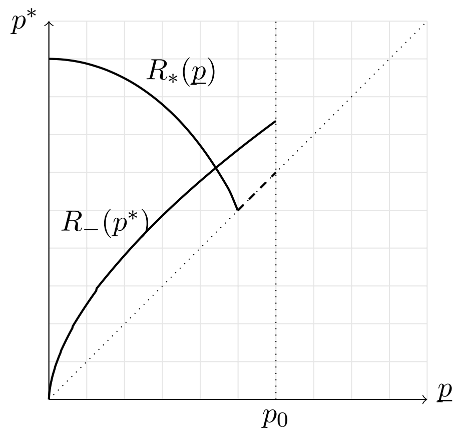
While firms with a rating that is too low are simply not visited at all, firms with high ratings enjoy queues that ensure they sell out. Moreover, these queues are longer at higher-rated firms in order that consumers remain indifferent. Thus, from an individual firm's perspective, profits $\pi(p) = \min\left\{\frac{g(p)}{f(p)}, \bar{\lambda}\right\}$ take the form of a step function:

$$\pi(p) = \begin{cases} \bar{\lambda} & \text{if } p \geqslant p^* \\ 0 & \text{if } p < p^* \end{cases} \tag{12}$$

This simple structure means that, all told, only three scalars – the thresholds $\underline{p}, p^*$ and the entry rate η – are required to fully characterize equilibrium strategies. In particular, proving the existence and uniqueness of equilibrium boils down to a pair of simple, scalar fixed-point problems. The basic idea is as follows: fix the entry rate η. Trace the locus $R^*(\underline{p})$ that yields the optimal p^* given a fixed $\underline{p}$, as well as the mirror image $R_-(p^*)$. By the uniqueness of both $\underline{p}$ in Proposition 2 and p^* in Proposition 9, these loci are both functions. I argue that they intersect precisely once.

First, an increase in p^* leaves firms worse off – on average, they make positive profits for a shorter fraction of their life-cycle. Hence, firms exit earlier and thus $\underline{p}$ must rise – $R_-(p^*)$ is upward sloping. Simply put, if consumers enjoy higher standards, fewer firms reap the rewards, depressing incentives. The reaction of consumers to firms' exit is a more technically challenging but equally illuminating problem. The key insight of my proof is to show that, when $\underline{p}$ rises, the ratings distribution $f(p)$ *is lower at every rating* p. In particular, fewer high quality firms remain active. Here we see the negative effect that exit induces on selection. In response, consumers are forced to lower their standards, as fewer firms remain in the right tail of the ratings distribution. Thus, $R^*(\underline{p})$ is downward sloping. Figure 1 demonstrates this fixed-point problem graphically. Finally, it remains to show that the value to an entrant $V(p_0)$ is decreasing in the entry rate, which follows from the fact that higher entry allows consumers to raise their

Figure 1: Reaction Functions



$R_*(\underline{p})$ optimal consumer threshold p^* given $\underline{p}$, $R_-(p^*)$ vice versa. Both are ill-defined when $\underline{p} \geqslant p_0$, as the market is empty

standards (p^* rises) and in turn depress firms' values.

Theorem 1. *There exists a unique stationary equilibrium, featuring: 1) Positive rates of both entry and exit, 2) sell-out profits and rapid feedback at established firms, 3) losses and slow feedback at struggling firms.*

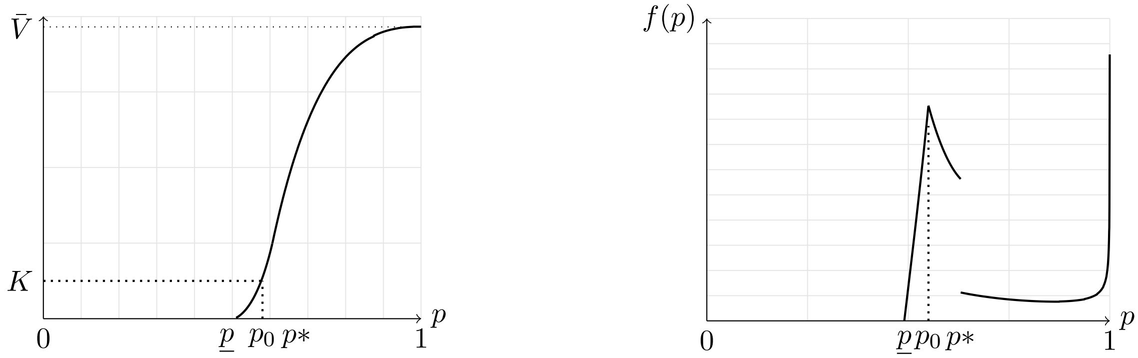
2.3 Equilibrium Features

A Tough Climb and a Fear of Falling – Learning from consumer reviews transforms the step profit function $\pi(p)$ into a smooth, S-shaped value function for firms, as shown in Figure 2. Intuitively, below the consumer threshold p^*, firms derive option value from their rating potentially rising via background learning, whereas above p^*, feedback simply leaves firms fearful that their rating might drop, losing their consumer base.

Lemma 4. *A firm's equilibrium value function $V(p)$ is S-shaped. Specifically, $V''(p) > 0$ for all $p \in [\underline{p}, p^*)$ and $V''(p) < 0$ for all $p \in [p^*, 1]$.*

Rapid exit stems from both the daunting task of slowly climbing the ratings ladder and the fearful prospect of rapidly slipping back down once climbed. Put differently, firms prefer a *fast-slow* profile of learning, which would make the climb easier and the fall less precipitous. Alas, in equilibrium, the profile is precisely the reverse, i.e. *slow-fast*. Here then, we begin see the misallocation of information

Figure 2: Firms' Value Function, Ratings Distribution



Left panel: Value function $V(p)$. Right panel: ratings distribution $f(p)$.

that occurs under full transparency, an insight that will prove invaluable when considering the platform's design problem in Section 3.

Fat Tails and Power-Law Ratings Distribution – Classical models of industry dynamics combine firm entry, exit and a stochastically evolving state variable to generate an endogenous equilibrium distribution of firms.[7] Empirical work has documented that the right tail of this distribution asymptotically follows a power law. Such a feature is shared by the ratings distribution $f(p)$ generated by this model. In fact, the tractability of the model allows me to solve for the distribution in closed-form:

Proposition 2.

Let

$$\gamma_0^f = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2\delta}{\epsilon}}, \qquad \gamma_1^f = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2\delta}{\epsilon + \bar{\lambda}}}$$

Then for $[\underline{p}, p^*]$, $f(p)$ is a linear combination of the functions

$$p^{-1-\gamma_0^f}(1-p)^{\gamma_0^f - 2} \quad \textit{and} \quad p^{\gamma_0^f - 2}(1-p)^{-1-\gamma_0^f},$$

for $[p^*, 1]$, $f(p)$ is a linear combination of the functions

$$p^{-1-\gamma_1^f}(1-p)^{\gamma_1^f - 2} \quad \textit{and} \quad p^{\gamma_1^f - 2}(1-p)^{-1-\gamma_1^f}$$

[7] Typically, the state is a measure of productivity. See for instance Hopenhayn (1992), Jovanovic (1982), Luttmer (2007), Atkeson et al. (2015). See Bar-Isaac (2003), Board and Meyer-ter-Vehn (2014) for examples where the state is a belief regarding product quality.

Full details are given in the Appendix.

Firms enter at a rating p_0, causing a kink at $f(p_0)$, and flow to either side depending on the reviews left by consumers. If $p^* > p_0$, then users of the platform do not visit new firms, and thus entrants must hope that the background flow of information ϵ is enough to push their rating above p^* so that they might make profits. Once above p^*, a firm's rating moves at a faster rate, since the added flow of consumer reviews creates a more precise feedback process. Figure 2 summarizes these features graphically.

This asymmetry in the flow-rate of information has a striking effect on f. Slow learning for the worst firms induces a steep left tail. Indeed, as background learning becomes negligible, firms below p^* face slim prospects for their rating to climb and thus exit rapidly. This outflow thins the left tail of the distribution, transferring weight to the right.

Corollary 1. *The ratings distribution exhibits a discontinuous drop at the consumer's threshold p^*, and has a fatter right tail than left.*

Cross-sectioning by Firm Age – While unnecessary for the determination of stationary equilibria, it is possible to analytically solve for the distribution of firms by age as well as ratings, $f(p,t), t \in [0,\infty)$. Several intuitive predictions can be extracted. Older firms exhibit a conditional distribution of output with higher mean - an immediate consequence of selection - but lower variance.[8] Exiting firms tend to be middle-aged; new entrants inevitably begin their ratings process bounded away from the exit threshold, while established incumbents are necessarily of high quality simply through selection, and thus have high ratings. Furthermore, cross-correlating firm age with the quantity of reviews received generates a further prediction that is almost immediate in a model with reputation formation and exit decisions: firms with many reviews tend to have a greater ratio of good to bad reviews than firms with few reviews.

Welfare – To prepare for the subsequent normative analysis, it is instructive to briefly discuss how one might make welfare comparisons across stationary equilibria of the model. The welfare of consumers is simple and unambiguous, as they are short-lived. Accounting for long-lived firms subject to entry and exit across time is less straightforward. However, due to the free entry condition imposed on firms, standard arguments exert that a suitable measure of social welfare at a stationary equilibrium is consumer

[8] Jovanovic (1982) shares these predictions.

welfare[9]:

$$\text{SW} \triangleq \int_{\underline{p}}^{1} p\Theta(p)g(p)dp + \eta\left[V(p_0) - K\right] = \text{CW} \tag{13}$$

Thus, a platform attempting to maximize either social or consumer welfare in the long-run will adopt the same policy.

3 Ratings Design

Recent studies show that, while many platforms simply aggregate reviews to construct product ratings, gains could be enjoyed from more carefully designed systems (Dai et al. (forthcoming)). Furthermore, some sites, such as Zomato, actively engage in so-called *ratings normalization*, whereby they transform a restaurant's rating as a function of the entire ratings distribution with the explicit intention of guiding consumer choice.[10] I interpret ratings design as the mapping from a firm's history of reviews into those reviews included when updating the rating. Given the complexity of solving for the full equilibrium fixed-point problem via the ratings distribution, little progress would be made at this level of generality, and so I will restrict attention to a class of rating policies that are both tractable to study and simple to implement.

Definition 2 (Ratings)**.** Given a cumulative review process X, a **rating policy** Y is a real-valued process that is progressively measurable w.r.t. $\mathbb{F}^x$. A rating policy Y is **simple** if there exists $r : [0,1] \to [0,1]$ such that:

(R_1) $\frac{1}{r}$ is locally integrable and square integrable, with:

$$dY_t = r(p_t)\lambda_t \theta dt + \sqrt{r(p_t)\lambda_t}\sigma dZ_t, \quad \text{ where } p_t = \mathbb{E}_t^y(\theta)$$

Denote simple policies by $\{r\}$. A simple policy $\{r\}$ involves **certification at** $\tilde{p}$ if for some $\tilde{p} \in (p_0, 1)$, $r(p) = 0$ for all $p \geqslant \tilde{p}$, is **all or nothing at** $\tilde{p}$ if $r(p) = \mathbb{I}_{p \geqslant \tilde{p}}$ for some $\tilde{p} \in (p_0, 1)$ and is **fully transparent** if $r(p) = 1$ for all $p \in [0,1]$. A simple policy $\{r\}$ is **sticky at** $\tilde{p} > p_0$ if $r(\tilde{p}) = 0$ and is not left continuous at p and **sticky at** $\tilde{p} < p_0$ if $r(p) = 0$ and is not right continuous at p.

[9]See Burdett and Menzio (2018), Olszewski and Wolinsky (2016) for discussions.

[10]See https://www.zomato.com/blog/simplifying-ratings-for-a-better-dining-experience.

Condition R_1 admits a simple, practical description. Simple policies prescribe a rate at which reviews are excluded from the firm's public review record. This rate – captured by the function r – can depend on the firms current rating in a fairly general manner. Certification policies certify firms once they reach a certain threshold – they are guaranteed to remain at this rating until they exit. All-or-nothing policies are a special case of these in which the platform posts all the reviews it receives for a firm prior to certification. Crucially, full transparency is simple (set $r(p) = 1$ for all p).

Some technical amendments are required in order to reconcile the formal definitions in Section 2 with sticky rating policies. For instance, if $\{r\}$ is sticky at p_1 then the ratings process exhibits a *sticky boundary* at p_1 – if a firm reaches p_1 then their rating remains there until they exit. Consequently, the distribution admits an atom at p_1. Since the details of these amendments are largely technical, I relegate them to Appendix.

I endow the platform with the ability to commit to a ratings policy. Practically speaking, it would be challenging to update a rating policy in light of firm-specific or even distributional evidence as it evolves.[11]

Definition 3 (Implementability)**.** If $\{r\}$ is not sticky, then $\{r\}$ is **implementable** if there exists a collection $E = \{V, J, g, f, \underline{p}, \eta\}$ such that equations (4), (6), (7), (9), (10) and the relevant conditions listed in Lemma 2 and Proposition 1 all hold. If $\{r\}$ is sticky, see Appendix A.2 for the relevant conditions. In this case, we say that $\{r\}$ **implements** E and that E is a (stationary) equilibrium w.r.t $\{r\}$. Let $\mathcal{E}(\lambda) = \{E : \{r\} \text{ implements } E\}$. Finally, a simple policy $\{r\}$ is **non-empty** if it implements an equilibrium E at which $\eta > 0$.

Thus, a policy is implementable if it supports a stationary equilibrium as previously defined. Empty policies are of course always dominated, since full transparency supports non-zero entry rates and thus strictly positive consumer welfare.

The central question in this section then is what can the platform achieve by designing a simple rating system? To begin with, since consumers are effectively solving the same problem, the allocation $g(p)$ retains its p^*-structure *at any equilibrium for any policy.*

[11] Yelp! stress that since their algorithm is automated, their staff cannot override the inclusion/exclusion of reviews from a firm's rating. See https://www.yelp-support.com/article/Why-would-a-review-not-be-recommended?l=en_US.

Lemma 5. *1. Let $\{r\}$ be non-empty with $E \in \mathcal{E}(r)$. Then there exists $p^*_E(r) \in (0,1)$ such that*

$$\lim_{p \nearrow p^*_E(r)} G(p) = 0 \quad \textit{and} \quad \Theta(p^*_E(r)) = 1, \Theta(p) < 1 \ \ \forall p > p^*_E(r)$$

Let $p^*(r) = \max_{E \in \mathcal{E}(r)} p^*_E(r)$

2. Consumer welfare at equilibrium E is given by $p^*_E(r)$.

To see that $p^*_E(r)$ equals consumer welfare, note that at this rating consumers are guaranteed service, and thus their expected value is simply $p^*_E(r)$. The indifference condition in equation (11) completes the proof. This argument holds irrespective of the ratings distribution, and thus for any simple policy. As such, the platform's optimization problem is remarkably simple to state:[12]

$$\underset{\{r\}}{\text{maximize}} \ \ p^*(r)$$

Crucially, an increase in p^* directly depresses firms' profits. Here then, we see the stark tension faced by the platform; the objective is to ensure as great a split of the surplus for consumers as possible, but, by doing so, this shrinks the surplus itself by depressing firms' incentives and thus reducing the quantity of highly-rated participating firms. The platform must then compensate firms through low-powered incentives; *choosing the policy that delivers firms' optimal feedback rates ensures that consumer welfare is maximized.*

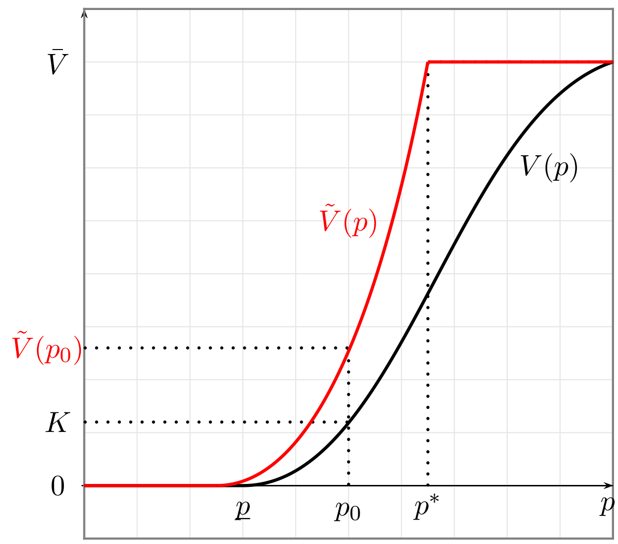
This transforms the computationally intractable problem of searching over policies while solving for their respect equilibrium conditions into a single-firm, dynamic information control problem, in the spirit of Moscarini and Smith (2001). Lemma 4 provides full guidance on what the solution to this control problem might be – a *fast-slow* feedback rate. The following result constitutes the main result of this section.

Theorem 2. *The optimal policy $\{r\}$ is all-or-nothing at $p^*(r)$.*

Figure 3 provides a related intuition. Start with the full transparency equilibrium $\mathcal{E} = \{V, f, p^*, J, \underline{p}, \eta\}$, and consider the value function $\tilde{V}$ under the policy proposed in Theorem 2, but evaluated at p^*. Lemma 4 tells us that $\tilde{V} > V$ point-wise, and so $\tilde{V}(p_0) > K$. A tâtonnement argument then implies that more

[12] Embedded in this expression is the assumption that were a policy $\{r\}$ to admit multiple equilibria ($|\mathcal{E}(r)| > 1$), the platform can select between them.

Figure 3: Optimal policy



Black lines: full transparency, with equilibrium strategies $\underline{p}, p^*$. Red lines: Counterfactual payoffs under optimal policy taking p^* as fixed. Crucially, $\tilde{V}(p_0) > V(p_0) = K$.

firms enter to erode these positive profits, relaxing capacity constraints and ultimately leading to an increase in p^*.

4 Extensions

4.1 Prices

Markets vary in the degree to which prices flexibly adjust in response to changes in ratings. Some markets exhibit secular price rigidity, whereas others exhibit price flexibility. For instance, markets for medical services are typically not cleared by prices that track ratings and popularity, and recent findings suggest that service markets such as restaurants and hotels also exhibit prices that only partially adjust to changes in ratings-driven demand.[13] In product markets however, prices tend to be more elastic to ratings. (Saeedi (2019)) Allowing firms to tailor prices to their evolving ratings may introduce intuitive equilibrium phenomena. For instance, firms may price below marginal cost when young or struggling, in

[13] See Lewis and Zervas (2016) for hotels, Andersen and Magruder (2012) for restaurants, Horton (2018) for evidence from AirB'nB, and the following link for a discussion on wait times in US healthcare: https://www.hhnmag.com/articles/6417-the-push-is-on-to-eliminate-hospital-wait-times.

order to attract additional consumers and thus higher feedback rates. In contrast, popular firms might charge higher prices in order to soak up the excess demand caused by their ratings success. Thus, it is of interest then to study ratings design in markets with either rigid and flexible prices.

I introduce competitive pricing into the model by employing the approach of the competitive search literature. In short, firms set prices as a function of their rating but compete across ratings via a consumer indifference condition similar to (11).

The firm's problem is now a hybrid control-stopping-time problem; in addition to choosing an exit threshold $\underline{p}$, the firm also solves a dynamic pricing problem. Of course, the complexity of this control problem in the current setting is that the pricing decision affects the firms present value not only through the usual demand schedule, but also through the informational rents that follow - that is, the SDE given in equation (2) now depends explicitly on prices via π_t. The firm's present value is now given by:

$$v(p) = \sup_{\tau^x, q \in \mathcal{Q}(p)} \mathbb{E}^{x,q} \left[\int_0^{\tau^x} e^{-(\rho+\delta)t} (q_t \pi_t - c) dt | p_0 = p \right], \tag{14}$$

subject to (2).[14] A simple modification to Assumption 2 to ensure gains from trade are guaranteed:

Assumption 3. $K < \frac{p_0 \bar{\lambda} - c}{\rho + \delta}$

Standard results in the theory of mixed stopping-control problems allow us to transform this expression into a tractable variational problem, itself a slightly extended version of the free boundary problem in Lemma 2.

Lemma 6. *Let the pair* $\{u(.), \underline{p}\}$ *for* $u \in \mathcal{C}^1([0,1])$ *and* $\underline{p} \in [0,1]$ *denote the variational inequality:*

$$\begin{aligned} \mathcal{A}_q\{u\} &= 0 \\ u(\underline{p}) &= 0 \\ u'(\underline{p}) &= 0 \\ u(p) &\geqslant 0 \quad \forall p \in [\underline{p}, 1] \\ u(p) &= 0 \quad \forall p \in [0, \underline{p}], \end{aligned} \tag{15}$$

[14] $\mathcal{Q}(p)$ denotes the set of processes progressively measurable with respect to $\mathbb{F}^x$ and such that equation (2) has a strong solution, given that the process $(p_t)_{t \geqslant 0}$ is initially at p.

where

$$\mathcal{A}_q\{u\} = \sup_{q\in\mathbb{R}} \left\{ q\pi(p,q) + (\pi(p,q)+\epsilon)\frac{p^2(1-p)^2}{2\sigma^2}u''(p) \right\} - c - (\rho+\delta)u(p) \tag{16}$$

Suppose $\{V(.), \underline{p}\}$ solves the problem (15), and that with respect to the process $(p_t)_{t\geqslant 0}$, the function $q(p)$ attains the supremum in (16), and that

$$\tau = \inf_s \{s \geqslant 0 : p_s \leqslant \underline{p}\} \tag{17}$$

Then the function $V(.)$ coincides with the value $v(.)$ in (14) and the pair $((q_t)_{t\geqslant 0}, \tau)$ attain the supremum in (14).

Thus, prices can be viewed as a function of firms' ratings and so the consumer's problem can yet again be written succinctly:

$$J = \max_{p\in[\underline{p},1], q\leqslant p} \left[(p-q)\,\Theta(p,q) \right], \tag{18}$$

Note that the indifference condition holds *for all incentive-compatible* prices, i.e. even those that are not charged by any firm on the equilibrium path. This embodies the *market utility (MU) assumption* commonly used in models of competitive search - were a firm to deviate in its posted price, the resulting queue-length it expects to find is as if the price were on-path.[15] A stationary equilibrium is thus defined as before, but with the addition of the policy $\{q\}$ as per (16).

The MU assumption, combined with the piecewise-linear matching, tells us that if firms have such price flexibility, congestion will cease to exist in equilibrium. Were a firm charging a price q and facing a lengthy queue, they would deviate to charge a higher price q' that partially alleviates congestion in a manner that keeps consumers happy, while clearly benefiting the firm. An immediate corollary is that equilibrium prices are affine in quality:

Lemma 7. *There exists $w \geqslant 0$ such that firms' equilibrium pricing policy $\{q\}$ satisfies $q(p) = p - w$. Furthermore, $\Theta(p) \leqslant 1$ for all $p \in [\underline{p}, 1]$. Equilibrium consumer welfare equals w.*

Having pinned down equilibrium prices as affine in ratings, the remaining choice for firms is whether or not to trade at these prices. That is, the service rate π can be viewed itself as the relevant control variable, taking prices as given. To understand this choice further, we can effectively re-write the HJB

[15] This refinement ensures that prices and thus equilibrium allocations correspond to competitive allocations. See Galenianos and Kircher (2012) for details.

equation governing incumbents' value functions:

$$
\begin{aligned}
V(p) &= \max_{q \in [\underline{q}, \bar{q}]} \left\{ \frac{1}{\rho+\delta} \left[q\pi(p,q) - c + (\pi(p,q)+\epsilon) \frac{p^2(1-p)^2}{2\sigma^2} V''(p) \right] \right\} \\
&= \max_{\pi \in [0, \bar{\lambda}]} \left\{ \frac{1}{\rho+\delta} \left[\left(p - w + \frac{p^2(1-p)^2}{2\sigma^2} V''(p) \right) \pi - c + \epsilon \frac{p^2(1-p)^2}{2\sigma^2} V''(p) \right] \right\}
\end{aligned} \tag{19}
$$

From this expression, it is clear that optimal service rates are bang-bang, i.e. there exists a $\tilde{p}$ such that $\pi(p) = \mathbb{I}_{p \geqslant \tilde{p}}$, and so firms' flow profit function is piece-wise linear, while of course satisfying the equilibrium market clearing condition:

$$\int_{\tilde{p}}^{1} g(p)dp = 1 \tag{20}$$

In summary, equilibrium strategies are then characterized by the scalars $\{\underline{p}, w, \tilde{p}, \eta\}$. This tractable structure allows me to derive the following features in closed-form:

Proposition 3. *Under competitive search pricing, there exists a unique stationary equilibrium, featuring: 1) positive rates of both entry and exit, 2) no congestion, 3) prices that increase with ratings, 4) pricing below marginal cost for some firms, and 5) globally convex value functions $V(p)$ for firms.*

Item 4) does not occur in models with exogenous information flows, e.g. Jovanovic (1982) – from equation 19, it is clear the supply decision depends not only on flow returns $p - w$ but also the option value of information $\frac{p^2(1-p)^2}{2\sigma^2}V''(p)$. Item 5), combined with the intuition acquired through Theorem 2, tells us that within the space of simple ratings, full transparency is optimal. Simply put, convexity implies that firms desire feedback at all ratings. However, by considering a broader set of policies, I will demonstrate that yet again simple censorship policies dominate full transparency.

Specifically, I consider a class of *private simple certification* policies. These are similar to simple certification policies, except the platform is now allowed to form private beliefs by monitoring all the reviews it receives. Thus, above the certification threshold $\tilde{p}$, the platform keeps all new reviews suppressed from public record, but if the firms true rating ever falls back below $\tilde{p}$, the platform starts posting new reviews again.

Definition 4. Y is a **private simple** rating policy if there exists $r : [0,1] \to [0,1]$ such that $\frac{1}{r}$ is locally integrable and square integrable, with

$$dY_t = r(p_t)\lambda_t \theta dt + \sqrt{r(p_t)\lambda_t}\sigma dZ_t, \quad \text{where } p_t = \mathbb{E}_t^x(\theta)$$

Private simple certification and all-or-nothing policies are defined analogously to Definition 2.

Proposition 4. *Under competitive search pricing, there exist private all-or-nothing ratings policies that dominate full transparency.*

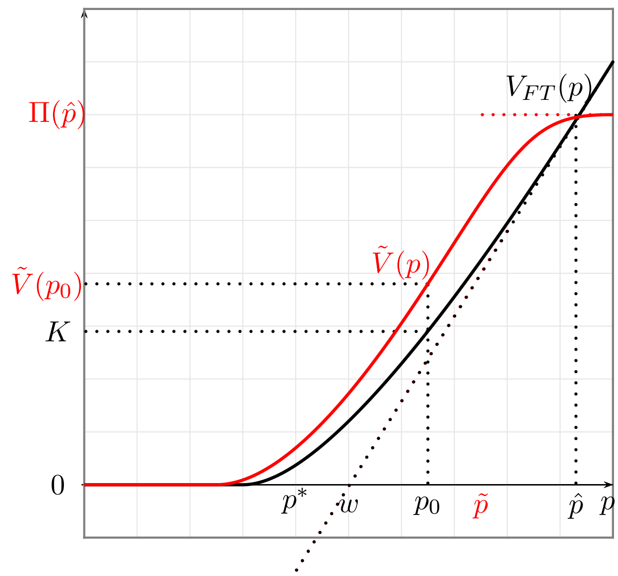
Whilst similar in appearance to Theorem 2, the qualitative mechanism behind Proposition 4 is quite distinct. Consider such a policy with associated certification threshold $\tilde{p}$. Firms with a (true) rating $p \geqslant \tilde{p}$ are effectively pooled together into one coarse category. Being unable to distinguish between them, consumers form an estimate of such firms ratings, given by $\mathbb{E}^x(p|p \geqslant \tilde{p})$. This estimate pins down the prices such firms command in equilibrium. Firms with (true) ratings just above $\tilde{p}$ command a price above their fully transparent benchmark, as $\tilde{p} < \mathbb{E}^x(p|p \geqslant \tilde{p})$. Conversely, firms with ratings $p > \mathbb{E}^x(p|p \geqslant \tilde{p})$ operate at a relative loss by being pooled with lower quality firms. The crucial insight here is that firms with lower ratings are (on average) younger. Thus, such policies exhibit between-firm transfers of surplus from well-established incumbents to young, less secure firms. By front-loading profits in this manner, these policies boost the present-discounted value of market entry, and thus leads to more firms entering. Classical arguments then show how this additional competitive pressure reduces equilibrium prices and so boosts consumer welfare. Figure 4 demonstrates this construction graphically.

Proposition 4 demonstrates that neither congestion nor S-shaped continuation values are necessary for the key insight from Theorem 2 – that the suppression of reviews can stimulate participation and provide improvements to consumer welfare – to hold. Furthermore, it shows how this insight extends to settings in which continuation values are convex in reputation, i.e. in markets with superstar returns or in which investment in capacity expansion or quality can convexify continuation values. (see Lucas (1978) and Board and Meyer-ter-Vehn (2014)) Taken together, Theorem 2 and Proposition 4 demonstrate two settings in which suppression can stimulate participation through qualitatively distinct channels.

4.2 Heterogeneous Consumers

In this section, I provide a heuristic description extending the model to allow for horizontally-differentiated consumer preferences. Specifically, a fraction $\alpha > 0$ of consumers derive expected utility $1-p$ from a firm with a rating of p, and the remaining consumers are as in Section 2. Call the former l-type and the latter r-type consumers. I call such markets α-segmented, and each group of consumers a *market segment*. One can think of this approach as a reduced-form for spatial preferences or taste shocks more generally.

Figure 4: Optimal policy under prices



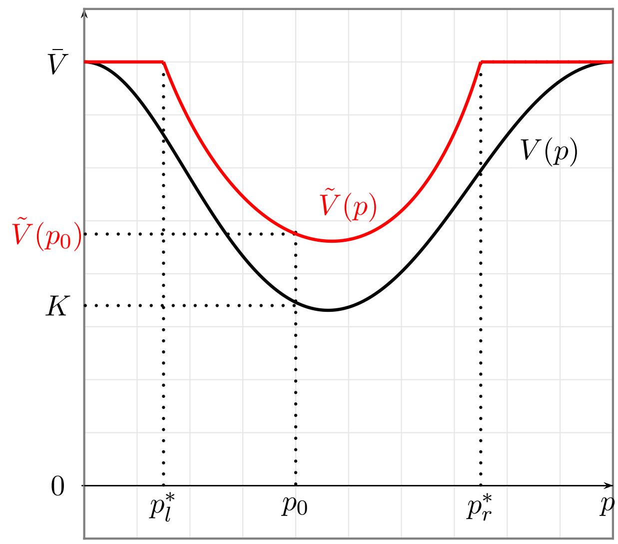
Black: full transparency, with prices $q(p) = p - w$. Red: counterfactual payoffs under all-or-nothing policy with certification at $\tilde{p}$ taking prices $q(p)$ as given and where $\hat{p} \triangleq \mathbb{E}^x(p|p \geqslant \bar{p})$ and $\Pi(\hat{p}) = \frac{\hat{p}-w-c}{\rho+\delta}$. Crucially, $\tilde{V}(p_0) > V_{FT}(p_0) = K$.

Firms now stand to make positive profits at both ends of the ratings distribution, capturing the distinct segments of the consumer market. Free entry ensures that some firms with middling ratings operate at a loss – now, the problem of a firm is to simply uncover which type they are as quickly as possible. Two features become rapidly apparent. First, consumer choice is now summarized by two thresholds, $0 < p_l^* < p_r^* < 1$ – l-type consumers visit firms with ratings below p_l^*, and r-type consumers visit firms above p_r^*. Second, no firms voluntary exit the market.[16]

As such, continuation values become *double S-shaped*, i.e. concave-convex-concave in ratings, and the optimal simple rating policy would now a *two-sided all-or-nothing* policy. The analysis would effectively be two identical copies of benchmark model, with the absence of exit as the only novel outcome of the co-existing market segments. This would justify the simplicity of the main analysis by showing how to a large extent markets with distinct segments can be analyzed by treating each segment in isolation. Figure 5 demonstrates this logic graphically.

[16] If $p_l^* \geqslant p_r^*$, then all firms would sell out forever, violating free entry. If firms exit at some rating $\underline{p}$, then these firms must not be serving consumers, violating consumer indifference – such exiting firms would be extremal in the distribution, and thus attract either l- or r-type consumers more than other incumbent firms.

Figure 5: Optimal policy with heterogeneous consumer preferences



Black lines: full transparency, with equilibrium strategies p_l^*, p_r^*. Red lines: Counterfactual payoffs under optimal policy taking p_l^*, p_r^* as fixed, with $\tilde{V}(p_0) > V(p_0) = K$.

4.3 Effort

The full suppression of reviews above p^* in Theorem 2 gives rise to moral hazard concerns – firms that reach the rating p^* would immediately slack off and exert low effort, knowing that they are guaranteed sell-out profits until attrition. Recent empirical findings (Hui et al. (2018a)) show that the entry-exit selection margin is far more sensitive to ratings design than moral hazard in seller quality provision. Nevertheless, in this section I consider the implications of allowing firms to make an observable effort choice that complements their underlying quality much as in classic career concerns models e.g. Hörner (2002). Suppose the firm chooses an observable effort level $a_t \in [0, \bar{a}]$ that induces flow cost of $Ca_t, C > 0$ and such that effort and quality are substitutes:[17]

$$dX_t = \lambda_t(\theta + a_t)dt + \sqrt{\lambda_t}\sigma dZ_t \tag{21}$$

[17]Whilst the effort choice is observable, I consider rating policies that do not condition on effort choices directly, thereby retaining elements of dynamic moral hazard models.

and thus the firm's present value becomes:

$$v(p) = \sup_{\tau^x, a \in \mathcal{Q}(p)} \mathbb{E}^{x,q}\left[\int_0^{\tau^x} e^{-(\rho+\delta)t}(\pi_t - Ca_t - c)dt | p_0 = p\right], \tag{22}$$

Similar arguments would yield an HJB equation of the form:

$$V(p) = \max_{a \in [a, \bar{a}]} \left\{ \frac{1}{\rho+\delta}\left[\pi(q) - Ca - c + (\pi(p)+\epsilon)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right]\right\} \tag{23}$$

The analysis combines features from Sections FT and Prices. For instance, equation 23 shows how effort linearizes flow payoffs to firms, while it now plays the role of prices in the consumer indifference equation:

$$J = \max_{p \in [\underline{p}, 1], a \in [0, \bar{a}]} \left[(p + a(p))\,\Theta(p)\right], \tag{24}$$

When marginal effort costs are high, firms exert effort at intermediate ratings with an effort schedule that decreases in ratings to ensure consumer indifference, but slack off at high ratings. This gives rise to a S-shaped value function for firms, and so again upper censorship policies would improve welfare. Effectively, equilibrium demand provides highly-rated firms with weak incentives to exert effort, and thus the platform is willing to forego these incentives in favor of the increased participation that censorship policies bring about. When marginal costs are low, all firms exert effort, and the analysis would resemble Section 4.1. These arguments form but a first step toward a complete analysis of moral hazard in such settings, which is reserved for future work.

4.4 Adverse Selection

My analysis assumes that firms do not know their own quality and learn alongside the market, as in Jovanovic (1982), Board and Meyer-ter-Vehn (2014), Ericson and Pakes (1995). This is meant to capture the notion that the suitability of its product within a market is not fully understood by the firm *ex ante*. With asymmetric information, firms could signal their quality through entry/exit choices, as in Bar-Isaac (2003), Atkeson et al. (2015). In particular, if each firm knew their type *ex ante*, free entry would dictate that only good firms enter. To see this, note that the continuation values for each type of firm now satisfy:

$$V_\theta(p) = \frac{1}{\rho+\delta}\left[\pi(p) - c + (\theta - p)V_\theta'(p) + \Sigma(p)V_\theta''(p)\right]$$

Since $V(p)$ is increasing, it is immediate that $V_1(p) > V_0(p)$ for all $p \in [0,1]$. Free entry would then entail only θ_h-type firms entering the market, and in particular no *ex-post* heterogeneity and a degenerate ratings distribution. Such a profile is clearly violated empirically, thus motivating the assumption that at least to some degree, firms do not fully know their quality *ex ante*.

5 Contribution and Related Literature

The main contribution of the paper is to two strands of research - the design of **recommendation/ratings systems** and **platform design**. It is the first analysis to study the role of information design in shaping industry dynamics through endogenous participation. Hörner and Lambert (2018) studies the design of ratings in order to incentivize a single firm to exert hidden effort and improve output quality. In their analysis, the arrival rate of information is independent of the firm's current rating, and thus abstracts from the cold start constraint central to the current paper. Furthermore, their model comprises a single firm, and thus the distributional concerns central to my analysis are absent. Hörner (2002) does study the interaction of competitive forces with firms effort choices and exit decisions, showing how competition can mitigate the inefficiencies that plague settings with career concerns and moral hazard. However, consumers do not learn socially, and the paper also abstracts from the implementation problem I study here. Che and Hörner (2018) and Kremer et al. (2014) examine the intertemporal informational externality that consumer choices generate, and thus also identify policies that can help alleviate the cold-start problem. They treat the range of products as exogenous, and thus abstract from firms' incentives. Also, by focusing on the single-product case, their consumers necessarily have an exogenous incentive-compatibility constraint, whereas in my setting this constraint is determined by endogenous participation. Goel and Thakor (2015) argues that coarse credit ratings might balance the need for transparency in financial markets against the need to alleviate moral hazard. The idea of diverting consumers from their optimal product choices in order to redistribute market power amongst competing producers is present in Hagiu and Jullien (2011), Yang (2018), Kovbasyuk and Spagnolo (2018) and Romanyuk and Smolin (forthcoming). The latter argues how congestion can occur naturally as a result of excess information, and thus argue how restricting information via upper censorship can alleviate this inefficiency, whereas in Kovbasyuk and Spagnolo (2018), discounting past observations from firm's ratings can be welfare-enhancing in the presence of evolving hidden quality. Both papers abstract from the entry/exit choices of firms. Indeed, that upper censorship plays a role in my analysis even in the absence of congestion (see section 4.1) highlights the importance of participation constraints in determining the value of information. The optimality of *upper censorship* disclosure policies is present in a number of recent papers in the persuasion literature (Romanyuk and Smolin (forthcoming), Yang (2018), Bloedel and Segal (2018), Kolotilin et al. (2017)). In Bloedel and Segal (2018), the result obtains due to costly information processing and thus the need to trade-off accuracy against fidelity. Finally, the provision of a coarse information policy by a monopolist echoes results from the **certification** literature (Lizzeri (1999), Biglaiser (1993)). More recently, Marinovic et al. (2018) extend these results to a dynamic setting with moral hazard, demonstrating that restricting the ability to voluntarily certify can improve incentives to invest in quality.

My analysis shares features of the literature on **collective experimentation** and **social learning**, wherein the central inefficiencies center on dynamic free-riding effects. Models of collective learning via Brownian diffusion processes can be found in Bolton and Harris (1999) and Bergemann and Välimäki (1997). Moscarini and Smith (2001) cast the sampling rate of information as a direct control variable. Beyond the classic works in the social learning literature (Bikhchandani et al. (1992), Banerjee (1992)), Acemoglu et al. (2018) also study in recent work the endogenous speed of learning in an observational learning framework. They argue that consumer selection drives both the accuracy and speed of feedback, and abstract from firms' incentives entirely. A related recent paper is Campbell et al. (2018), which to the best of my knowledge, is the only other paper to endogenize the underlying quality distribution in an otherwise standard social learning setting. Their model features social learning between consumers connected on a network, rather than via a platform, and firm entry, and abstracts from information design question my normative theory addresses.

A classic literature in industrial organization describes how informational asymmetries can pose a **barrier to entry** for late arriving firms (Schmalensee (1982), Bagwell (1990), Grossman and Horn (1988)). These papers highlight how superior information regarding product quality can endow incumbent firms with a first-mover advantage, leading to inefficient entry choices, even with price setting. Schmalensee

(1982) outlines a two-period model, in which the incumbent and entrant are of identical quality. Bagwell (1990) extends this analysis by allowing the incumbent to be of lower quality than the entrant, showing how inefficiencies still prevail. Grossman and Horn (1988) focuses on a moral hazard margin in the choice of quality in a trade setting. However, none of these papers incorporate the social learning feature of information diffusion that my analysis hinges on, nor do they consider ratings design as a method for stimulating entry.

My model also contributes to the literature on **firm dynamics**, in which firms make entry and exit decisions that are governed by an evolving state process. Classical models such as Hopenhayn (1992) and Luttmer (2007), Jovanovic (1982) and Board and Meyer-ter-Vehn (2014) also center around predictions regarding cross-sectional firm distributions and dynamic hazard-rates. Technically, Luttmer (2007) also builds upon the theory of *resetting processes*, whereby firms exit at some state z and arrive at some other state $z' > z$. Closely related is the work by (Atkeson et al., 2015), who study a competitive market with adverse selection and firm dynamics. Again, the asymmetry in learning rates is not present, nor is the information design problem – their policy instrument is the entry cost. Furthermore, the market structure is quite different: there, a firm's demand function is linear in its current rating and increasing in the overall mass of active firms. Bar-Isaac (2003) involves a firm that solves a similar stopping problem, the main differences being the endogenously generated flow profit function as well as variable learning rates. In a recent paper, Kuvalekar and Lipnowski (2018) study the firing decision of a firm over a worker with no monetary transfers, and show how the worker's equilibrium choices involves signal jamming as a means of avoiding rapid exit. Also related is Hörner (2002), who shows how market forces can alleviate standard moral hazard incentive concerns by embedding a standard career-concerns problem into a competitive market with entry and exit decisions. There, the assumptions that consumers are forward-looking and learn privately as well as that one firm can serve the entire market gives rise to severe switching effects not present in my analysis.

The baseline model itself contributes to the theory of **mean-field games**, models which comprise a backward equation (here, the HJB equation governing firms' continuation values) and a forward equation (here, the Fokker-Planck equation governing the ratings distribution). Gabaix et al. (2016) and Moscarini (2005) both involve the derivation of boundary conditions similar to those outlined in Proposition 1. Mine is the first study to explicitly embed an information design problem into such an environment. This confluence poses new technical challenges, which I solve using elements of both stopping and control theory, as well as the theory of so-called *oscillating diffusion processes* - see le Gall (1985) and Keilson and Wellner (1978).

Finally, the empirical predictions offered in Section 2.3 speak to the **empirical** literature on consumer reviews. Previous studies that attempt to back out quality from reviews include Dai et al. (forthcoming), (Andersen and Magruder, 2012), Li and Hitt (2008), Chevalier and Mayzlin (2006). For instance, Dai et al. (forthcoming) assumes that the underlying process governing firm quality is mean zero, capturing as a reduced-form the martingale property of belief updating under Bayesian inference. Horton (2018) and Fradkin (2018) document the presence of severe congestion externalities due to capacity constraints in online platforms. Relatedly, Lewis and Zervas (2016) document how increased demand due to reputational effects does not fully pass through into higher prices, implying the co-existence of congestion and price discrimination (see Section 4.1 for a discussion in the present context). In recent work, Luca and Luca (2018) document a negative correlation between a firm's average rating and the probability of exit. Hui et al. (2018b) demonstrate how, following a tightening of the certification standards at an online platform, entry initially increases but then levels out in the long-run, while the quality distribution of new entrants exhibits a higher mean and thicker right tail. However, theirs is a setting with severe adverse selection concerns. Of particular note is a recent working paper by Li et al. (2018), who document an interesting policy intervention that temporarily allowed sellers at an online trading platform to pay for reviews via rebates. They demonstrate how new entrants used the service more than established players, consistent with the results presented here.

6 Conclusion

This paper explored whether consumer reviews can form a *barrier to entry* for new firms. To this end, I built a tractable, equilibrium model in which firms of heterogeneous quality make entry and exit decisions, and whose quality is gradually revealed via consumer reviews. A *rating platform* controls the inclusion of these reviews into a firm's rating. Ratings guide consumer search and thus provide firms with incentives to remain active. Under a regime of *full transparency*, consumers flock to the highest-rated firms. This demand renders firms' continuation values S-shaped as a function of ratings. Thus, while struggling firms covet feedback and established firms dislike feedback, *precisely the reverse* profile obtains. Turning to design, I modeled the platform as maximizing consumer welfare via its control of the rating process. The central result was that optimal rating design involves *upper censorship* — the exclusion of reviews from established firm's ratings — as a means of making the task of surmounting the ratings hill less daunting, thus stimulating participation.

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A Formalities

A.1 Local Time for Ratings Process

In this section, I derive the general expression of the *local time* of a firm's rating process. See Stokey (2008) for a comprehensive treatment.

Let the ratings process solve the SDE given by (2) defined over $[0,1]$. This process is formally an *oscillating diffusion process*, for which the existence of a strong solution was shown in le Gall (1985) and explicit formulae for the density and occupation times were derived in Keilson and Wellner (1978). In particular, p_t converges to either 0 or 1 when unstopped, but achieves neither state in finite time. Now, let τ be the $\mathbf{F}^x$-measurable stopping time defined as $\tau = T(\underline{p})$, where $T(p) \triangleq \min\{t : p_t = p\}$ is the first hitting time of the ratings process for a given level. Since $T(1) = \infty$, it follows that $\tau = T(\underline{p}) \wedge T(1)$. Taking expectations of Theorem 3.7 in Stokey (2008), the firm's value function can be re-written:

$$
\begin{aligned}
v(\hat{p}) &= \mathbb{E}^x \left[\int_0^\tau e^{-(\rho+\delta)t} (\lambda(p_t) - c) dt | p_0 = \hat{p} \right] \\
&= \mathbb{E}^x \left[\int_0^1 l(p; \hat{p}, \tau, \rho + \delta)(\lambda(p) - c) dp \right] \\
&= \int_{\underline{p}}^1 L(p; \hat{p}, \underline{p}, 1, \rho + \delta)(\lambda(p) - c) dp
\end{aligned}
$$

where $l(p; \hat{p}, \tau, \rho + \delta)$ is the discounted local time for the ratings process defined as in Theorem 3.4 in Stokey (2008),

$$
L(p; \hat{p}, a, b, \rho + \delta) = \mathbb{E}^x \left[l(p; \hat{p}, T(a) \wedge T(b), \rho + \delta) \right] \quad \text{for } a, b \in [0, 1]
$$

The purpose of this appendix is to derive closed-form solutions for the function $L(p; \hat{p}, a, b, \rho + \delta)$ where a, b are such that $\tilde{\lambda}(p) = \lambda \in \mathbb{R}_+$. To this end, let $T = T(a) \wedge T(b)$, and define

$$
w(\hat{p}) = \mathbb{E}^x \left[\int_0^T e^{-rt} g(p_t) dt | p_0 = \hat{p} \right] \tag{A.1}
$$

for some $g \in \mathcal{C}^2([0,1])$. Standard arguments yield that w solves the second-order ODE

$$
rw(p) = g(p) + \Sigma(p) w''(p), \tag{A.2}
$$

where $\Sigma(p) = \frac{\lambda}{2\sigma^2} p^2 (1-p)^2$ and subject to the boundary conditions $w(a) = w(b) = 0$. The two independent general solutions to equation (A.2) are

$$w_1(p) = p^{\gamma}(1-p)^{1-\gamma}$$
$$w_2(p) = p^{1-\gamma}(1-p)^{\gamma}$$

where $\gamma = \frac{1}{2} + \sqrt{\frac{1}{2} + \frac{2r\sigma^2}{\lambda}}$. A particular solution for (A.2) is

$$\bar{w}(p) = \sum_{i=1}^{2} \xi_i(p) w_i(p)$$

where $\xi_i(p), i = 1, 2$ are functions to be determined. Conjecture that

$$\sum_{i=1}^{2} \xi_i'(p) w_i(p) = 0$$

, in which case

$$\Sigma(p) \sum_{i=1}^{2} \xi_i(p) w_i(p) = -g(p)$$

and thus for the conjecture to be true, it must be that

$$\xi_1'(p) = \frac{w_2(p)g(p)}{\mathbb{W}(w_1, w_2)(p)\Sigma(p)}, \quad \xi_2'(p) = \frac{w_1(p)g(p)}{\mathbb{W}(w_1, w_2)(p)\Sigma(p)},$$

where $\mathbb{W}(w_1, w_2) = w_1 w_2' - w_2 w_1'$ is the Wronskian. Direct calculation (omitted algebra) results in

$$\xi_1'(p) = \frac{2\sigma^2}{(1-2\gamma)\lambda} p^{-1-\gamma}(1-p)^{\gamma-2} g(p), \quad \xi_2'(p) = \frac{2\sigma^2}{(1-2\gamma)\lambda} p^{\gamma-2}(1-p)^{-1-\gamma} g(p),$$

and thus

$$\xi_1(p) = \frac{2\sigma^2}{(1-2\gamma)\lambda} \int_a^p z^{-1-\gamma}(1-z)^{\gamma-2} g(z) dz, \quad \xi_2'(p) = \frac{2\sigma^2}{(1-2\gamma)\lambda} \int_p^b z^{\gamma-2}(1-z)^{-1-\gamma} g(z) dz$$

Combining the expressions for the general and particular solutions for $w(p)$, and imposing the boundary conditions

$w(a) = w(b) = 0$ gives rise to the full expression:

$$
\begin{aligned}
w(p) &= \xi_1(p)w_1(p) + \xi_2(p)w_2(p) - \psi(p)\xi_2(a)w_2(a) - \Psi(p)\xi_1(b)w_1(b) \\
&= \frac{2\sigma^2}{(1-2\gamma)\lambda}\left[\int_a^p \left(\frac{p}{z}\right)^{\gamma}\left(\frac{1-p}{1-z}\right)^{1-\gamma}\frac{1}{z(1-z)}g(z)dz\right. \\
&\quad + \int_p^b \left(\frac{p}{z}\right)^{1-\gamma}\left(\frac{1-p}{1-z}\right)^{\gamma}\frac{1}{z(1-z)}g(z)dz \\
&\quad - \psi(p)a^{1-\gamma}(1-a)^{\gamma}\int_a^b z^{\gamma-2}(1-z)^{-1-\gamma}g(z)dz \\
&\quad \left. -\Psi(p)b^{\gamma}(1-b)^{1-\gamma}\int_a^b z^{-1-\gamma}(1-z)^{\gamma-2}g(z)dz\right]
\end{aligned}
\tag{A.3}
$$

where

$$
\psi(p) = \frac{w_1(p)w_2(b) - w_1(b)w_2(p)}{w_1(a)w_2(b) - w_1(b)w_2(a)}, \qquad \Psi(p) = \frac{w_1(p)w_2(a) - w_1(a)w_2(p)}{w_1(b)w_2(a) - w_1(a)w_2(b)}
$$

To compute $L(p; \hat{p}, a, b, r)$, set $g(p) = 1$ and take derivatives of equation (A.3):

Proposition A.1. *Suppose $a, b \in [0,1]$ such that $\tilde{\lambda}(p) = \lambda \in \mathbb{R}_+$ for all $p \in [a,b]$. Let $L(p; \hat{p}, a, b, r)$ denote the expected discounted local time for the ratings process p between levels a, b, with initial rating $\hat{p}$ and discount rate r. Then*

$$
\begin{aligned}
L(p; \hat{p}, a, b, r) = \frac{2\sigma^2}{(1-2\gamma)\lambda}&\left[\Xi(p;\hat{p}) - \psi(\hat{p})a^{1-\gamma}(1-a)^{\gamma}p^{\gamma-2}(1-p)^{-1-\gamma}\right. \\
&\left. -\Psi(\hat{p})b^{\gamma}(1-b)^{1-\gamma}p^{-1-\gamma}(1-p)^{\gamma-2}\right]
\end{aligned}
\tag{A.4}
$$

where

$$
\Xi(p;\hat{p}) = \begin{cases} \left(\frac{\hat{p}}{p}\right)^{\gamma}\left(\frac{1-\hat{p}}{1-p}\right)^{1-\gamma}\frac{1}{p(1-p)} & \text{if } p \leqslant \hat{p} \\ \left(\frac{\hat{p}}{p}\right)^{1-\gamma}\left(\frac{1-\hat{p}}{1-p}\right)^{\gamma}\frac{1}{p(1-p)} & \text{if } p \geqslant \hat{p} \end{cases}
$$

A.2 Ratings design: sticky policies

In this section, I extend the defining conditions for a stationary equilibrium with respect to sticky ratings policies. Modifications are required for both the ratings distribution f and the value function V.

The ratings process thus attains such points in finite time, with such p being termed *sticky boundaries*. First, I claim that it is without loss to consider only policies $\{r\}$ for which there exists at most one p_1 such that $\{r\}$ is sticky at p_1, and furthermore that $p_1 > p_0$. To wit, suppose $p_1 \in \mathcal{O}$ and $p_1 < p_0$. Either the firm is selling out at p_1, in which case Lemma 5 implies it sells out at all $p \geqslant p_1$ thus violating the free entry condition, or it makes a loss, in which case it exits at p_1 and so $f(p_1) = 0$. Next, suppose there exist p_1, p_2 both greater than p_0 such that $p_1 < p_2$ and $r(p_1) = r(p_2) = 0$. Then $f(p) = 0$ for all $p > p_1$ and thus the value that r takes at p_2 is irrelevant.

Thus, if a policy $\{r\}$ takes value 0 at $p_1 > p_0$, the ratings distribution admits an atom at p_1, as firms are trapped at p_1 until they experience exogenous death at rate δ. By using Lemma B.1, we derive an additional boundary condition to account for this atom. Formally, the ratings distribution is defined by the pair $\{f, F_1\}$ that satisfy:

(F_1) $f(\underline{p}) = 0$

(F_2) $\Sigma(p_0)[f'_-(p_0) - f'_+(p_0)] = \eta$

(F_3) $\Sigma(1)f(1) = 0$

(F_4) $\frac{\partial}{\partial p}\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, p_0) \cup (p_0, 1])$

(F_5) $\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, 1])$

(F_6) $\delta[\int_{\underline{p}}^1 f(p)dp + F_1] + \Sigma(\underline{p})f'_+(\underline{p}) = \eta$

(F_7) $\delta F_1 + [\Sigma(p_1)f(p_1)]'_- = 0$

Turning to the value function V, it is defined as before for all $p < p_1$, but leveling discontinuously to $\bar{V}$ at p_1. Formally, amend the variational inequality in Lemma 2 to apply to the pair $\{u(.), \underline{p}\}$ for $u \in \mathcal{C}^1$ on $[0, p_1]$, $u \in \mathcal{C}$ on $[0, 1]$ and $\underline{p} \in [0, 1]$, and which must satisfy conditions (4) of Lemma 2, condition (5) for all $p \in [\underline{p}, p_1)$ and:

$$\mathcal{A}_1\{u\} = \pi(p) - c - (\rho + \delta)u(p)$$

for all $p \in [p_1, 1]$.

B Proofs

B.1 Lemma 2

Note that the conditions of Rüschendorf and Urusov (2008) hold, since the SDE for the state variable p has no drift coefficient and a diffusion coefficient given by $\lambda(p)p(1-p)$, which is locally bounded since $\lambda(p) \in [0, \bar{\lambda} + \epsilon]$ for all $p \in [0, 1]$. Their Theorem 2.1 implies that the unique value function satisfying (3) belongs to $\mathcal{C}^1([0, 1])$, and solves (4), and that $\underline{p}$ is unique. Finally, boundedness of the value function follows from the Extreme Value Theorem.

B.2 Proposition 1

Here, I derive the boundary conditions for the Fokker-Planck equation (7), drawing upon techniques from the adjoint theory of differential operators to derive boundary conditions. See Gabaix et al. (2016) for an economic application of this approach, and Gardiner (2009) more generally.

Let $X \in \mathbb{B}(\mathbb{R})$ and for two functions $u, v \in \mathcal{L}^2(X)$, define their inner product as $\langle u, v \rangle = \int_X u(x)v(x)dx$. Further, for an operator $\mathcal{A}$, the *adjoint operator* is defined as $\mathcal{A}^*$ such that $\langle u, \mathcal{A}v \rangle = \langle \mathcal{A}^* u, v \rangle$. For a diffusion process Y satisfying $dY_t = a(Y,t)dt + b(Y,t)dW_t$ for an appropriately defined Wiener process W with constant hazard-rate of death δ, the *infinitesimal operator* is given by

$$\mathcal{A}_b\{u\}(x,t) = a(x,t)\frac{\partial u}{\partial x}(x) + \frac{1}{2}b(x,t)\frac{\partial^2 u}{\partial x^2}(x) - \delta u(x)$$

Finally, the operator

$$\mathcal{J}\{u\}(x,t) = a(x,t)u(x) - \frac{\partial}{\partial x}(b(x,t)u(x))$$

denotes the *mass flux*, i.e. for $S \subset \mathbb{R}$, the integral $\int_{\partial S} \mathcal{J}\{f\}(x,t)$ measures the total mass crossing the boundary of S per unit time (to see this, integrate the non-stationary version of (7) directly and use the Fundamental Theorem of Calculus).

Thus, for the ratings process defined by the SDE (2),

$$(\mathcal{A}_b\{u\})(p,t) = \frac{1}{2}\Sigma(p)\frac{\partial^2 u}{\partial p^2} - \delta u(p) \tag{B.1}$$

$$\mathcal{J}\{u\}(p,t) = -\frac{\partial}{\partial p}\left[\Sigma(p)u(p)\right] \tag{B.2}$$

where $\Sigma(p)$ is as defined in Lemma 2. Standard results in statistical mechanics then imply that the transition measures form a root of the adjoint operator to (B.1).

Remark 1. *For the process Y defined above, the stationary distribution satisfies $\mathcal{A}_f f = 0$, where $\mathcal{A}_f = \mathcal{A}_b^*$.*

This verifies that f must solve (7). It remains to derive the boundary conditions for f. To do so, we state the relevant boundary conditions for the equation $\mathcal{A}\{u\} = 0$. Standard results tell us that the operator $\mathcal{A}$ is well-behaved, i.e. that solutions to $\mathcal{A}\{u\} = 0$ lie in $\mathcal{C}^2$. We use this and the adjoint relation in Remark 1 to transform these into conditions on f.

Lemma B.1. *Let $\tilde{\mathcal{D}}$ denote the set of discontinuities of f, and let $\mathcal{D} = \tilde{\mathcal{D}} \cup \{\underline{p}, p_0, 1\}$. Then*

$$\int_{\mathcal{D}} \left[u(p)\mathcal{J}\{f\}(p) + f(p)\Sigma(p)\frac{\partial u}{\partial p}\right] dp = 0 \tag{B.3}$$

for all $u \in \mathcal{C}^2([0,1])$.

Proof.

$$\begin{aligned}\langle \mathcal{A}^* f, u\rangle &= -\int_0^1 u\left[(\Sigma f)'' - \delta f\right] dp \\ &= \int_0^1 \left[(u(\Sigma f)')' - \Sigma f u' - \delta f u\right] dp + \int_{\mathcal{D}} u(\Sigma f)' - f\Sigma u_p dp \\ &= \int_0^1 f\left[\Sigma u'' - \delta u\right] dp + \int_{\mathcal{D}} u\left[(\Sigma f)' - f\Sigma u_p\right] dp \\ &= \langle f, \mathcal{A}u\rangle - \int_{\mathcal{D}} u\left[\mathcal{J}\{f\} + f\Sigma u_p\right] dp\end{aligned}$$

where the second equality follows by the Fundamental Theorem of Calculus. The result then follows by Remark 1. ☐

Armed with Lemma B.1, I now derive conditions 1 - 5 of Proposition 1 (condition 6 is derived later separately from an aggregate conservation of probability principle).

1. $f(\underline{p}) = 0, \Sigma(1)f(1) = 0$

   We have that $f(\underline{p})\Sigma(\underline{p})u' = 0$ for all u. Since $\underline{p} > 0$, it follows that $\Sigma(\underline{p}) \neq 0$, and hence $f(\underline{p}) = 0$. This is the standard "attainable boundary"condition (see Feller (1954) for a comprehensive classification of boundary conditions for diffusion processes over bounded intervals). The second condition follows similarly. This is a "natural boundary"condition, as $p = 1$ cannot be attained in finite time.

2. $\Sigma(p_0)\left[f'_-(p_0) - f'_+(p_0)\right] = \eta$, $\frac{\partial}{\partial p}\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, p_0) \cup (p_0, 1])$, $\Sigma(p)f(p) \in \mathcal{C}^1([\underline{p}, 1])$

   An inflow rate of η yields the boundary condition $[\Sigma(p_0)u(p_0)]_-^+ = \eta$ on any u that solves $\mathcal{A}_f\{u\} = 0$. Furthermore, since u and u' are continuous, this implies that:

$$\begin{aligned}\left[\mathcal{J}\{f\}(p_0)\right]_-^+ &= \eta \\ -\left[\Sigma(p_0)f'(p_0) + \Sigma'(p_0)f(p_0)\right]_-^+ &= \eta \\ \Sigma(p_0)\left[f'_-(p_0) - f'_+(p_0)\right] &= \eta\end{aligned}$$

   Intuitively, the condition states that the total outflow of mass from p_0, given by $[\mathcal{J}\{f\}(p_0)]_-^+$, must equal the total inflow η. Note that as a consequence, it also implies that f is continuous at p_0. The continuity conditions on Σf and $(\Sigma f)'$ also follow from that fact that u and u' are both continuous.

Finally, condition 6 is derived by imposing that the total mass of firms is constant in a stationary equilibrium, i.e.

$$\frac{d}{dt}\int_{\underline{p}}^1 f(p,t)dp = 0 \tag{B.4}$$

Direct substitution of the Fokker-Planck equation into equation (B.4) implies that

$$[(\Sigma f)']_{\mathcal{D}} - \delta \int f dp = 0$$
$$-\Sigma(\underline{p})f'_{+}(\underline{p}) + \eta - \delta \int f dp = 0$$

where all other terms in $[(\Sigma f)']_{\mathcal{D}}$ vanish due to continuity of $(\Sigma f)'$ and $f(\underline{p}) = 0$.

B.3 Lemma 3

First, note that in the presence of background learning at rate $\epsilon > 0$, the stationary distribution must admit a density with full support on $[\underline{p}, 1]$. To obtain indifference in (9), if must be that for any rating visited in equilibrium, congestion must (weakly) occur, i.e. $\frac{g(p)}{\lambda f(p)} \geqslant 1$ for all $p \in \text{supp}(g)$. For if not, take $p_1, p_2 \in \text{supp}(g), p_1 \neq p_2$ such that $\frac{g(p_1)}{\lambda f(p_1)} < 1, \frac{g(p_2)}{\lambda f(p_2)} < 1$. The consumer then obtains an expected payoff of p_i by choosing to consume at p_i, and hence cannot be indifferent, since $p_1 \neq p_2$. This verifies that $\pi(p) \in \{0, \bar{\lambda}\}$. A similar argument verifies that $\pi(p)$ is increasing, and hence there exists $p^* \in [0, 1]$ such that $\pi(p) = 0$ for $p < p^*$ and $\pi(p) = \bar{\lambda}$ for $p \geqslant p^*$. This also establishes the conjectured form of $g(p)$. Finally, that $p^* > 0$ is a simple consequence of maintaining indifference.

Finally, I argue that in equilibrium, $p^* > \underline{p}$. First, since $p^* > 0$, the qualifier in Lemma 2 obtains, and thus $\underline{p} > 0$ in equilibrium. Suppose instead that $p^* \leqslant \underline{p}$. Then $\pi(p) = \bar{\lambda}$ for all $p \in [\underline{p}, 1]$, and it is then readily shown that the firm's value satisfies $V(p) = \bar{V}$ for all $p \in [\underline{p}, 1]$, violating the boundary condition $V(\underline{p}) = 0$.

B.4 Theorem 1

The outline of the proof is as follows:

1. Fix the entry rate η
   (a) Solve explicitly for f for arbitrary $\underline{p}, p^*$.
   (b) Solve explicitly for $(V, \underline{p})$ for arbitrary p^*.
   (c) Recast $\underline{p}, p^*$ as a pair of reaction correspondences, $R_-(p^*), R_*(\underline{p})$.
   (d) Prove that these correspondences are functions, with precisely one intersection.
2. Prove that the value of entry $V(p_0)$ is decreasing in the entry rate, with $V(p_0) = K$ always obtaining for some unique entry rate $\eta \in (0, \infty)$.

B.4.1 Computing the Ratings Distribution

The entry rate η will be assumed fixed until Section B.4.5 later in the proof. As such, f is solved for separately over three regions, the boundaries of which depend on the equilibrium value of p^*. In all cases, conditions 4 and 5 boil down to:

$$\left[\frac{\partial}{\partial p}\Sigma(p^*)f(p^*)\right]_-^+ = 0$$
$$[\Sigma(p^*)f(p^*)]_-^+ = 0$$

Case 1: $p^* > p_0$

The general solution to the Fokker-Planck equation (7) is obtained from the derivations outlined in Appendix A.1:

$$f(p) = \begin{cases} c_0^1 p^{-1-\gamma_0^f}(1-p)^{\gamma_0^f-2} + c_0^2 p^{\gamma_0^f-2}(1-p)^{-1-\gamma_0^f} & \text{for } p \in [\underline{p}, p_0] \\ c_1^1 p^{-1-\gamma_0^f}(1-p)^{\gamma_0^f-2} + c_1^2 p^{\gamma_0^f-2}(1-p)^{-1-\gamma_0^f} & \text{for } p \in [p_0, p^*) \\ c_2^1 p^{-1-\gamma_1^f}(1-p)^{\gamma_1^f-2} + c_2^2 p^{\gamma_1^f-2}(1-p)^{-1-\gamma_1^f} & \text{for } p \in [p^*, 1] \end{cases}$$

where γ_0^f, γ_1^f are as per the statement of the proposition. For $x, y \in [\underline{p}, 1]$, define:

$$\begin{aligned}
\alpha_i^1(x) &= x^{-1-\gamma_i^f}(1-x)^{\gamma_i^f-2} \\
\alpha_i^2(x) &= x^{\gamma_i^f-2}(1-x)^{-1-\gamma_i^f} \\
\psi_i^1(x,y) &= \int_x^y p^{-1-\gamma_i^f}(1-p)^{\gamma_i^f-2}dp \\
&= \int_{\frac{x}{1-x}}^{\frac{y}{1-y}} \left(\frac{z}{1+z}\right)^{-1-\gamma_i^f}\left(\frac{1}{1+z}\right)^{\gamma_i^f-2} dz \\
&= \int_{\frac{x}{1-x}}^{\frac{y}{1-y}} z^{-1-\gamma_i^f}(1+z)dz \\
&= \frac{1}{\gamma_i^f}\left[\left(\frac{x}{1-x}\right)^{-\gamma_i^f} - \left(\frac{y}{1-y}\right)^{-\gamma_i^f}\right] + \frac{1}{\gamma_i^f-1}\left[\left(\frac{x}{1-x}\right)^{1-\gamma_i^f} - \left(\frac{y}{1-y}\right)^{1-\gamma_i^f}\right] \\
\psi_i^2(x,y) &= \int_x^y p^{\gamma_i^f-2}(1-p)^{-1-\gamma_i^f}dp \\
&= \frac{1}{1-\gamma_i^f}\left[\left(\frac{x}{1-x}\right)^{\gamma_i^f-1} - \left(\frac{y}{1-y}\right)^{\gamma_i^f-1}\right] - \frac{1}{\gamma_i^f}\left[\left(\frac{x}{1-x}\right)^{\gamma_i^f} - \left(\frac{y}{1-y}\right)^{\gamma_i^f}\right]
\end{aligned}$$

Since $\gamma_1^f > 1$, condition 3 immediately implies that

$$c_2^2 = 0$$

Some (omitted) algebra then allows me to transform the five remaining boundary conditions into a system of five independent equations for the five remaining undetermined coefficients from the boundary conditions.

Lemma B.2. *The coefficients* $\mathbf{c} = [c_0^1 \; c_0^2 \; c_1^1 \; c_1^2 \; c_2^1]^T$ *solve the linear algebraic system* $\mathbf{Ac} = \mathbf{b}$*, where*

$$\mathbf{A} = \begin{bmatrix} \alpha_0^1(\underline{p}) & \alpha_0^2(\underline{p}) & 0 & 0 & 0 \\ \frac{\partial \alpha_0^1}{\partial p}(p_0) & \frac{\partial \alpha_0^2}{\partial p}(p_0) & -\frac{\partial \alpha_0^1}{\partial p}(p_0) & -\frac{\partial \alpha_0^2}{\partial p}(p_0) & 0 \\ 0 & 0 & \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^1(p^*) & \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^2(p^*) & -\frac{\partial}{\partial p}\Sigma_1(p^*)\alpha_1^1(p^*) \\ 0 & 0 & \Sigma_0(p^*)\alpha_0^1(p^*) & \Sigma_0(p^*)\alpha_0^2(p^*) & -\Sigma_1(p^*)\alpha_1^1(p^*) \\ \delta\psi_0^1(\underline{p}, p_0) + \Sigma_0(\underline{p})\frac{\partial \alpha_0^1}{\partial p}(\underline{p}) & \delta\psi_0^2(\underline{p}, p_0) + \Sigma_0(\underline{p})\frac{\partial \alpha_0^2}{\partial p}(\underline{p}) & \delta\psi_0^1(p_0, p^*) & \delta\psi_0^2(p_0, p^*) & \delta\psi_1^1(p^*, 1) \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ \eta \\ 0 \\ 0 \\ \eta \end{bmatrix}$$

Case 2: $p^* < p_0$

The general solution to the Fokker-Planck equation (7) now becomes

$$f(p) = \begin{cases} c_0^1 p^{-1-\gamma_0^f}(1-p)^{\gamma_0^f - 2} + c_0^2 p^{\gamma_0^f - 2}(1-p)^{-1-\gamma_0^f} & \text{for } p \in [\underline{p}, p^*] \\ c_1^1 p^{-1-\gamma_1^f}(1-p)^{\gamma_1^f - 2} + c_1^2 p^{\gamma_1^f - 2}(1-p)^{-1-\gamma_1^f} & \text{for } p \in [p^*, p_0) \\ c_2^1 p^{-1-\gamma_1^f}(1-p)^{\gamma_1^f - 2} + c_2^2 p^{\gamma_1^f - 2}(1-p)^{-1-\gamma_1^f} & \text{for } p \in [p_0, 1] \end{cases}$$

As before, $c_2^2 = 0$, and so coefficients $\mathbf{c} = [c_0^1 \; c_0^2 \; c_1^1 \; c_1^2 \; c_2^1]^T$ solve $\mathbf{Ac} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} \alpha_0^1(\underline{p}) & \alpha_0^2(\underline{p}) & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \alpha_1^1}{\partial p}(p_0) & \frac{\partial \alpha_1^2}{\partial p}(p_0) & -\frac{\partial \alpha_1^1}{\partial p}(p_0) \\ \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^1(p^*) & \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^2(p^*) & -\frac{\partial}{\partial p}\Sigma_1(p^*)\alpha_1^1(p^*) & -\frac{\partial}{\partial p}\Sigma_1(p^*)\alpha_1^2(p^*) & 0 \\ \Sigma_0(p^*)\alpha_0^1(p^*) & \Sigma_0(p^*)\alpha_0^2(p^*) & -\Sigma_1(p^*)\alpha_1^1(p^*) & -\Sigma_1(p^*)\alpha_1^2(p^*) & 0 \\ \delta\psi_0^1(\underline{p}, p^*) + \Sigma_0(\underline{p})\frac{\partial \alpha_0^1}{\partial p}(\underline{p}) & \delta\psi_0^2(\underline{p}, p^*) + \Sigma_0(\underline{p})\frac{\partial \alpha_0^2}{\partial p}(\underline{p}) & \delta\psi_0^1(p^*, p_0) & \delta\psi_0^2(p^*, p_0) & \delta\psi_1^1(p_0, 1) \end{bmatrix}$$

and $\mathbf{b}$ is as in the previous case.

B.4.2 Computing the Firm's Value Function

By Lemma 2, we know that the firm's value function belongs to $\mathcal{C}^1([0,1])$. Thus, the boundary conditions for the associated variational problem reduce to:

$$V(\underline{p}) = 0, \quad V'(\underline{p}) = 0, \quad V_-(p^*) = V_+(p^*), \quad V(1) = \frac{\bar{\lambda} - c}{\rho + \delta}$$

The general and particular solutions to equation (5) can be found by applying the derivation obtained in Appendix A.1:

Proposition B.1. *Let*

$$\gamma_0^v = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2(\rho+\delta)}{\epsilon}}; \qquad \gamma_1^v = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2(\rho+\delta)}{\epsilon + \bar{\lambda}}}$$

Then firms' value function given by

$$V(p) = \begin{cases} c_1^0 p^{1-\gamma_1^v}(1-p)^{\gamma_1^v} + \frac{\bar{\lambda}-c}{\rho+\delta} & \text{if } p \geqslant p^* \\ c_0^0 p^{1-\gamma_0^v}(1-p)^{\gamma_0^v} + c_0^1 p^{\gamma_0^v}(1-p)^{1-\gamma_0^v} - \frac{c}{\rho+\delta} & \text{if } p < p^* \end{cases} \tag{B.5}$$

where the coefficients $\begin{bmatrix} c_0^0 & c_0^1 & c_1^0 \end{bmatrix}' = \mathbf{c_v}$ *solve the linear algebraic system* $\mathbf{\Theta_v c_v} = \mathbf{b_v}$, *where*

$$\mathbf{\Theta_v} = \begin{bmatrix} \left[\frac{1-\gamma_0}{\underline{p}} - \frac{\gamma_0}{1-\underline{p}}\right] \underline{p}^{1-\gamma_0}(1-\underline{p})^{\gamma}_0 & \left[\frac{\gamma_0}{\underline{p}} - \frac{1-\gamma_0}{1-\underline{p}}\right] \underline{p}^{\gamma_0}(1-\underline{p})^{1-\gamma_0} & 0 \\ {p^*}^{1-\gamma_0}(1-p^*)^{\gamma_0} & {p^*}^{\gamma_0}(1-p^*)^{1-\gamma_0} & {p^*}^{1-\gamma_1}(1-p^*)^{\gamma_1} \\ \left[\frac{1-\gamma_0}{p^*} - \frac{\gamma_0}{1-p^*}\right] {p^*}^{1-\gamma_0}(1-p^*)^{\gamma_0} & \left[\frac{\gamma_0}{p^*} - \frac{1-\gamma_0}{1-p^*}\right] {p^*}^{\gamma_0}(1-p^*)^{1-\gamma_0} & -\left[\frac{1-\gamma_1}{p^*} - \frac{\gamma_1}{1-p^*}\right] {p^*}^{1-\gamma_1}(1-p^*)^{\gamma_1} \end{bmatrix}$$

and

$$\mathbf{b_v} = \begin{bmatrix} 0 \\ \frac{\bar{\lambda}}{\rho+\delta} \\ 0 \end{bmatrix}$$

and the threshold value $\underline{p}$ *solves*

$$c_0^0 \underline{p}^{1-\gamma_0}(1-\underline{p})^{\gamma_0} + c_0^1 \underline{p}^{\gamma_0}(1-\underline{p})^{1-\gamma_0} = \frac{c}{\rho+\delta} \tag{B.6}$$

noting that the coefficients c_0^0, c_0^1 *implicitly depend on* $\underline{p}$.

B.4.3 Reaction Functions

In light of Lemmas 2 and 3, equilibrium strategies are fully summarized by the pair of scalars$\{\underline{p}, p^*\}$.

Let $R_-(p^*)$ solve (B.6) with respect to $\underline{p}$ and $R_*(\underline{p})$ solve (10) subject to (11). That is, $R_-(p^*)$ computes optimal exit thresholds, fixing p^*, while $R_*(\underline{p})$ computes optimal demand thresholds p^* fixing $\underline{p}$.

Lemma B.3. *1. Both* $R_-(p^*)$ *and* $R_*(\underline{p})$ *are functions on* $[0,1]$. $R_-(p^*) \in \mathcal{C}^1([0,1])$, *while* $R_*(\underline{p}) \in \mathcal{C}([0,1])$.

2. For all $p^* > 0$, $R_-(p^*) < p^*$. $R_-(0) = 0$.

3. $R_*(0) > 0$.

Proof. 1. That $R_-(p^*)$ is single-valued is a consequence of the uniqueness of $\underline{p}$ in Lemma 2. To show that $R_*(\underline{p})$ is single-valued, note that it is easily verified that for profile $g(p)$ to solve (10), p^* must satisfy the equation:

$$p^* = \frac{\bar{\lambda}}{B} \int_{p^*}^{1} p f(p; p^*) dp \tag{B.7}$$

where $f(p; p^*)$ is used to denote the explicit dependence of f on p^*. It remains to prove that a solution to (B.7) exists, and is unique.

Suppose that $p^* > p_0$ (a similar argument will apply in the case $p^* < p_0$). Then (B.7) reduces to

$$\begin{aligned} p^* &= \frac{\bar{\lambda}}{B} \int_{p^*}^{1} p f(p; p^*) dp \\ &= \frac{\bar{\lambda}}{B} \int_{p^*}^{1} c_2^1(p^*) p^{-\gamma_1^f} (1-p)^{\gamma_1^f - 2} dp \\ &= \frac{\bar{\lambda}}{B} c_1^2(p^*) [\frac{(1-p^*)^{\gamma_1^f - 1} p^{*1-\gamma_1^f}}{\gamma_1^f - 1}] \end{aligned}$$

Consider the function $h(x) = c_1^2(x)(1-x)^{\gamma_1^f - 1} x^{1-\gamma_1^f} \equiv c_1^2(x) j(x)$. By the final part of the proof of Lemma 3, $h(0) > 0$, and $h(1) = 0$. Since $\gamma_1^f > 1$, $j(x)$ is decreasing in x. I now prove that $c_1^2(x)$ is decreasing in x.

To do so, I re-write the system of equations governing f, with the goal of splitting the system in two parts, defined by p_0. This allow direct computation of derivatives with respect to p^*. To this end, re-write the boundary conditions $\Sigma(p_0)[f'_-(p_0) - f'_+(p_0)] = \eta$ as $\Sigma(p_0) f'_-(p_0) = \alpha$ and $-\Sigma(p_0) f'_+(p_0) = \eta - \alpha$ for some $\alpha \in \mathbb{R}$. Thus, f solves the forward equation (7) on $[\underline{p}, p_0)$ subject to the conditions $f(\underline{p}) = 0$ and $\Sigma(p_0) f'_-(p_0) = \alpha$, and on $(p_0, 1]$ subject to $\Sigma(1) f(1) = 0$ and $-\Sigma(p_0) f'_+(p_0) = \eta - \alpha$. The constant α is then pinned down by the aggregate condition 6 in Proposition 1.

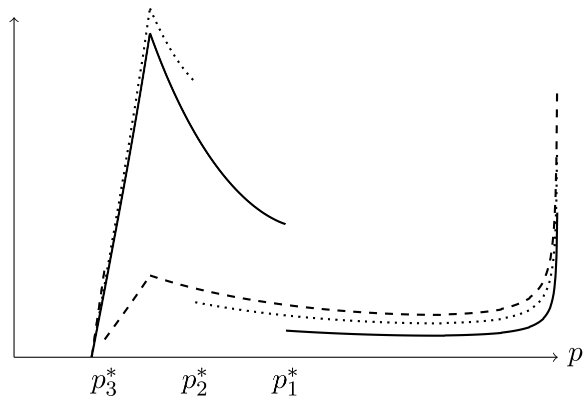
On $[\underline{p}, p_0)$, f is determined by the two coefficients $[c_0^1 c_0^2]^T$ that solve the system:

$$\begin{bmatrix} \alpha_0^1(\underline{p}) & \alpha_0^2(\underline{p}) \\ \frac{\partial \alpha_0^1}{\partial p}(p_0) & \frac{\partial \alpha_0^2}{\partial p}(p_0) \end{bmatrix} \begin{bmatrix} c_0^1 \\ c_0^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \tag{B.8}$$

while on $(p_0, 1)$, f is determined by the three coefficients $[c_1^1 c_1^2 c_2^2]^T$ that solve the system:

$$\begin{bmatrix} \frac{\partial \alpha_0^1}{\partial p}(p_0) & \frac{\partial \alpha_0^2}{\partial p}(p_0) & 0 \\ \frac{\partial}{\partial p} \Sigma_0(p^*) \alpha_0^1(p^*) & \frac{\partial}{\partial p} \Sigma_0(p^*) \alpha_0^2(p^*) & \frac{\partial}{\partial p} \Sigma_1(p^*) \alpha_1^1(p^*) \\ \Sigma_0(p^*) \alpha_0^1(p^*) & \Sigma_0(p^*) \alpha_0^2(p^*) & \Sigma_1(p^*) \alpha_1^1(p^*) \end{bmatrix} \begin{bmatrix} c_1^1 \\ c_1^2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} \eta - \alpha \\ 0 \\ 0 \end{bmatrix} \tag{B.9}$$

Figure 6: Ratings Density $f(p)$ for fixed $\underline{p}$ as p^* varies, $p_1^* > p_2^* > p_0 > p_3^*$



Direct computation then yields that c_2^2 is strictly decreasing in p^*, regardless of the value of α. In summary, we have shown that $h(0) > 0, h(1) = 0$ and $h(x)$ is strictly decreasing. Hence, by the Intermediate Value Theorem, there exists a solution to (B.7) (uniqueness obtains since h is injective), denoted p^*. If $p^* \geqslant \underline{p}$. Then set $R_*(\underline{p}) = p^*$. If $p^* < \underline{p}$, the unique profile that solves (9) subject to (10) is $g(p) = \frac{Bpf(p)}{\int_{\underline{p}}^1 pf(p)dp}$ and thus $R_*(\underline{p}) = \underline{p}$. Finally, continuous differentiability of $R_-(p^*)$ follows from applying the Implicit Function Theorem to (B.6).

2. For the first part, see the final part of the proof of Lemma 3. $R_-(0) = 0$ then follows by applying the Sandwich Theorem.

3. Let $f_0(;, p^*)$ denote the ratings distribution under no exit, for a given p^* [18]. It is readily shown that $f_0(;, p^*)$ is strictly positive almost everywhere for all $p^* \in [0, 1]$, and hence there exists $\epsilon > 0$ such that the RHS of equation (B.7) evaluated with respect to f_0 must be greater than ϵ.

□

Lemma B.4. *1.* $\frac{\partial R_-(p^*)}{\partial p^*} > 0$ *for all* $p^* \in [0, 1]$

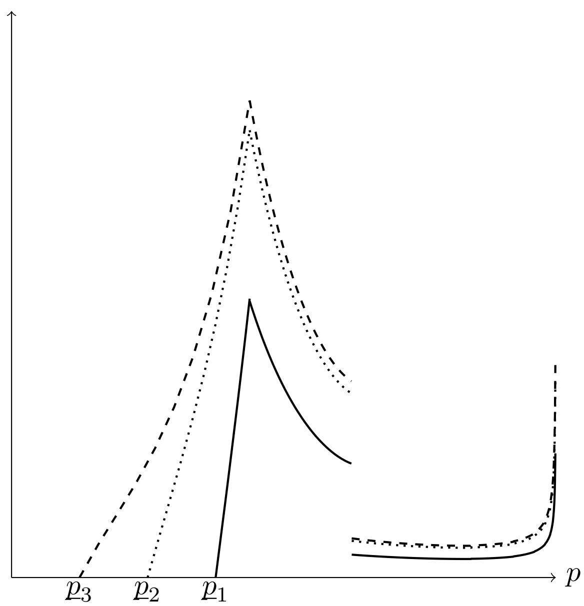
2. There exists $\tilde{p} \in (0, p_0)$ *such that* $\frac{\partial R_*(\underline{p})}{\partial \underline{p}} < 0$ *for all* $\underline{p} \leqslant \tilde{p}$ *and* $R_-(p^*) = \underline{p}$ *for all* $\underline{p} \geqslant \tilde{p}$

Proof. 1. Take $p_1^* < p_2^*$. Then $\lambda(p; p_1^*) > \lambda(p; p_2^*)$ for all $p \in (p_1^*, p_2^*]$ and $\lambda(p; p_1^*) = \lambda(p; p_2^*)$ everywhere else. Hence, by equation (3), a firm's value is strictly decreasing in p^*. The result then follows immediately from the conditions (4).

2. I first show that as the exit threshold decreases, more firms exist *at all ratings*, i.e. if $\underline{p}_1 < \underline{p}_2$, then $f(p, \underline{p}_1) > f(p, \underline{p}_2)$ for all $p > \underline{p}_1$. The result then follows immediately by expression (B.7) - a higher $\underline{p}$ makes the RHS smaller, and hence p^* must decline to maintain equality.

[18]Technically, I must adjust the boundary conditions appropriately, replacing condition 1 with the condition $\Sigma(0)f(0) = 0$ and amending condition 6 to read $\delta \int_0^1 f(p)dp = \eta$.

Figure 7: Ratings Density $f(p)$ for fixed p^* as $\underline{p}$ varies, $\underline{p}_1 > \underline{p}_2 > \underline{p}_3$



I utilize the decomposition of f introduced in the proof of Lemma B.3 part 1. Take the system (B.8). Direct computation yields that both c_0^1, c_0^2 are decreasing in $\underline{p}$. Thus, $f_-(p_0)$ is decreasing in $\underline{p}$. Now consider the system (B.9). Recall that f is continuous at p_0, as obtained in the proof of Proposition 1. As such, the boundary condition $-\Sigma(p_0)f'_+(p_0) = \eta - \alpha$ can be replaced by the simpler condition $f_+(p_0) = f_-(p_0)$, transforming the system into:

$$\begin{bmatrix} \alpha_0^1(p_0) & \alpha_0^2(p_0) & 0 \\ \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^1(p^*) & \frac{\partial}{\partial p}\Sigma_0(p^*)\alpha_0^2(p^*) & \frac{\partial}{\partial p}\Sigma_1(p^*)\alpha_1^1(p^*) \\ \Sigma_0(p^*)\alpha_0^1(p^*) & \Sigma_0(p^*)\alpha_0^2(p^*) & \Sigma_1(p^*)\alpha_1^1(p^*) \end{bmatrix} \begin{bmatrix} c_1^1 \\ c_1^2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} f_-(p_0) \\ 0 \\ 0 \end{bmatrix} \tag{B.10}$$

From this expression, it is then immediate that c_1^1, c_1^2, c_2^2 are all increasing in $f_-(p_0)$ and hence decreasing in $\underline{p}$.

□

B.4.4 Fixed point

Combining the various results of the previous section yields a simple fixed point problem for the pair of functions R_- and R_*.

Lemma B.5. *For fixed $\eta > 0$, there exists a unique tuple $\{V, f, J, \underline{p}, p^*\}$ that solve (4), (7), (9), (10), with g defined as in Lemma 3.*

Proof. $R_-^{-1}(p)$ is strictly increasing on $[0, \tilde{p}]$ with $R_-^{-1}(0) = 0, R_-^{-1}(\tilde{p}) > \tilde{p}$, while $R_*(p)$ is strictly decreasing on $[0, \tilde{p}]$ with $R_*(0) > 0$ and $R_*(\tilde{p}) = \tilde{p}$. The uniqueness of the pair $\underline{p}, p^*$ follows from the Intermediate Value Theorem..

The uniqueness of V, J, f then follows from Lemma 2, Lemma B.3 and the derivation in Section B.4.1. □

B.4.5 Entry

To summarize, I have shown that for a fixed entry rate η, firms' exit and consumers' demand thresholds are uniquely determined, in turn uniquely determining the ergodic ratings distribution. I now complete the proof of Theorem 1 by endogenizing η and proving that a unique η closes the system. Note that η effects an incumbent's value only indirectly through its effect on p^*. Let $p^*(\eta), V(p_0;\eta)$ denote these relationships.

Lemma B.6. *$f(p;\eta)$ is homogeneous of degree 1 in η, and hence $p^*(\eta)$ is strictly increasing.*

Proof. By Lemma 2, the coefficients determining f solve $\mathbf{Ac} = \eta\mathbf{1}$. Thus, using equation (B.7),

$$p^*(\eta) = \frac{\eta\bar{\lambda}}{B}\int_{p^*}^{1} pf(p;p^*)dp \tag{B.11}$$

Implicit differentiation yields

$$\frac{dp^*}{d\eta} = \underbrace{\left[\frac{\bar{\lambda}}{B}\int_{p^*}^{1} pf(p;p^*)dp\right]^{-1}}_{>0}\underbrace{\left[1 - \frac{\eta\bar{\lambda}}{B}\frac{d}{dp^*}\left[\int_{p^*}^{1} pf(p;p^*)dp\right]\right]}_{>0}$$

□

as the integral is positive and strictly decreasing in p^* - see Lemma B.3.

Lemma B.7. *1. $V(p_0;\eta)$ is strictly decreasing in η*

2. $\lim_{\eta\to 0} V(p_0;\eta) = \frac{\bar{\lambda}-c}{\rho+\delta}$

3. $\lim_{\eta\to\infty} V(p_0;\eta) = 0$

Proof. 1. Follows from Lemma B.6 and Lemma B.4.

2. First, note that as $\eta \to 0$, $p^* \to 0$. This follows from the fact that the integrand in (B.11) is bounded above for an entry rate of 1. The remainder of the proof is as follows. Take p^* small, and consider a slightly higher rating. This rating is also small, and hence the local evolution of the process is arbitrarily slow. As such, the firm's value is approximately equal to the discounted flow payoff of selling out. The result then follows from monotonicity of the firm's value function.

   Formally, set $p^* = \epsilon > 0$ and $\hat{p} = 2\epsilon$. I will show that the local time converges to a Dirac mass at $\hat{p}$ with measure bounded above by $\frac{1}{r}$ as $\epsilon \to 0$. By Proposition A.1, we have that

$$\begin{aligned}
L(\hat{p}; \hat{p}, \hat{p}-\epsilon, \hat{p}+\epsilon, r) &= \frac{2\sigma^2}{(1-2\gamma)\lambda\hat{p}(1-\hat{p})} - \psi(\hat{p})(\hat{p}-\epsilon)^{1-\gamma}(1-\hat{p}+\epsilon)^{\gamma}\hat{p}^{\gamma-2}(1-\hat{p})^{-1-\gamma} \\
&\quad - \Psi(\hat{p})(\hat{p}+\epsilon)^{\gamma}(1-\hat{p}-\epsilon)^{1-\gamma}\hat{p}^{-1-\gamma}(1-\hat{p})^{\gamma-2} \\
&= \frac{2\sigma^2}{(1-2\gamma)\lambda\epsilon(1-2\epsilon)} - \psi(2\epsilon)\epsilon^{1-\gamma}(1-\epsilon)^{\gamma}(2\epsilon)^{\gamma-2}(1-2\epsilon)^{-1-\gamma} \\
&\quad - \Psi(2\epsilon)(3\epsilon)^{\gamma}(1-3\epsilon)^{1-\gamma}(2\epsilon)^{-1-\gamma}(1-2\epsilon)^{\gamma-2} \\
&\geqslant \frac{2\sigma^2}{(1-2\gamma)\lambda\epsilon(1-2\epsilon)} - \epsilon^{1-\gamma}(1-\epsilon)^{\gamma}(2\epsilon)^{\gamma-2}(1-2\epsilon)^{-1-\gamma} \\
&\quad - (3\epsilon)^{\gamma}(1-3\epsilon)^{1-\gamma}(2\epsilon)^{-1-\gamma}(1-2\epsilon)^{\gamma-2} \\
&= \frac{2\sigma^2}{(1-2\gamma)\lambda\epsilon(1-2\epsilon)} + A\epsilon^{-1}(1-2\epsilon)^{-1}, \qquad \text{for some } A \in (0,\infty) \\
&\geqslant \frac{2\sigma^2}{(1-2\gamma)\lambda\epsilon(1-2\epsilon)} \\
&\to \infty \qquad \text{as } \epsilon \to 0
\end{aligned}$$

where the inequality on the third line is valid since $|\psi(p)|, |\Psi(p)| \leqslant 1$. On the other hand, for any $a, b \in (0,1)$, $\int_a^b L(p; \hat{p}, a, b, r)dp$ is clearly bounded above by $\frac{1}{r}$ - set $g(p) = 1$ in equation (A.1) and note that $T = T(a) \wedge T(b) < \infty$. Hence

$$\begin{aligned}
V(\hat{p}) &= \int_{\underline{p}}^{1} L(p; \hat{p}, \underline{p}, 1, \rho+\delta)(\lambda(p)-c)dp \\
&\geqslant \int_{\hat{p}-\epsilon}^{\hat{p}+\epsilon} L(p; \hat{p}, \underline{p}, 1, \rho+\delta)(\lambda(p)-c)dp \\
&\to \int_{\hat{p}-\epsilon}^{\hat{p}+\epsilon} \frac{\delta(\hat{p}-p)}{\rho+\delta}(\lambda(p)-c)dp \qquad \text{as } \epsilon \to 0 \\
&= \frac{\bar{\lambda}-c}{\rho+\delta}
\end{aligned}$$

Since $p_0 > \hat{p}$ for sufficiently small ϵ, the result follows from the monotonicity of $V(p)$.

3. Follows by a similar argument to 2.

□

In light of Definition 2, the uniqueness of η follows by the Intermediate Value Theorem:

Lemma B.8. *There exists a unique η such that $V(p_0; \eta) = K$*

B.5 Theorem 2

The following lemma will be used throughout the proof. Its proof is a direct application of the derivation in, for instance, Appendix A of Karatzas and Wang (2001) or more generally Section 2.6 in Karatzas and Shreve (1998).

Lemma B.9. *Let the stochastic process $(z_t)_{t\geqslant 0}$ solve the SDE:*

$$dz_t = \sigma(z_t)\sqrt{\lambda_t}dW_t, \tag{B.12}$$

on the interval $[0,1]$, $(\sigma\sqrt{\lambda})^{-1}$ is locally integrable and square integrable and where $(W_t)_{t\geqslant 0}$ is a standard Wiener process. Let $\Lambda(z_0)$ denote the set of processes progressively measurable with respect to $\mathbb{F}^z$ that take values in some bounded interval $[\underline{\lambda}, \bar{\lambda}] \subset \mathbb{R}_+$ such that equation (B.12) has a strong solution, and where the process $(z_t)_{t\geqslant 0}$ takes an initial value z_0. Let f be locally integrable and square integrable, and

$$v(z_0) = \sup_{\tau_z, \lambda\in\Lambda(z_0)} \mathbb{E}^z\left[\int_0^{\tau^z} e^{-(\rho+\delta)t} f(z_t)dt | z(0) = z_0\right] \tag{B.13}$$

Finally, let the tuple $\{V(.), \underline{z}, \lambda(.)\}$ denote the variational problem:

$$\begin{aligned}
\mathcal{A}_z\{V\} &= 0 \\
V(\underline{z}) &= 0 \\
V'(\underline{z}) &= 0 \\
V(z) &\geqslant 0 \quad \forall z \in [\underline{z}, 1] \\
V(z) &= 0 \quad \forall z \in [0, \underline{z}],
\end{aligned} \tag{B.14}$$

where

$$\mathcal{A}_z\{u\} = \max_{\lambda\in[\underline{\lambda},\bar{\lambda}]}\left\{\frac{1}{2}\sigma^2(z)\lambda u''(z)\right\} + f(z) - (\rho+\delta)u(z)$$

Suppose $\{V(.), \underline{z}, \lambda(.)\}$ solves the problem (B.14), and that with respect the process $(z_t)_{t\geqslant 0}$, the function $\lambda(z)$ is given by:

$$\lambda(z) = \begin{cases} \bar{\lambda} & \text{if } u''(z) \geqslant 0 \\ \underline{\lambda} & \text{if } u''(z) \leqslant 0 \end{cases} \tag{B.15}$$

and that

$$\tau = \inf\{t \geqslant 0 : z_t \leqslant \underline{z}\} \tag{B.16}$$

Then the function $V(.)$ coincides with the value (B.13) and the pair $((\lambda_t)_{t\geqslant 0}, \tau)$ attain the supremum in (B.13).

Take an arbitrary non-empty policy $\{r_1\}$. I claim that $p^*(r) \geqslant p^*(r_1)$. Suppose not. Under $\{r_1\}$, the value function for a firm is given by:

$$v(p) = \sup_{\tau^x} \mathbb{E}^y \left[\int_0^{\tau^x} e^{-(\rho+\delta)t} (\bar{\lambda}\mathbb{I}_{p_t \geqslant p^*(r_1)} - c) dt | p(0) = p \right] \tag{B.17}$$

subject to the SDE:

$$dp_t = \frac{\sqrt{r_1(p_t)\lambda(p_t)}}{\sigma} p_t(1-p_t) d\tilde{Z}_t,$$

and the initial condition $v(p_0) = K$. where Y_t is the cumulative review process for the policy $\{r_1\}$. By Lemma 2, this value coincides with the function $V_1(p)$, where:

$$V_1(p) = \frac{1}{\rho+\delta}\left[\bar{\lambda}\mathbb{I}_{p \geqslant p^*(r_1)} - c + \lambda(p)\frac{p^2(1-p)^2}{2\sigma^2}V_1''(p)\right], \quad V_1(\underline{p}(r_1)) = V_1'(\underline{p}(r_1)) = 0,$$

where $\underline{p}(r_1) \in (0,1)$ such that $\tau_\lambda = \inf\{\tau \geqslant 0 | p_t = \underline{p}(r_1)\}$ solves (B.17). Let the pair $(\tilde{V}(p), \tilde{\underline{p}})$ solve:

$$\tilde{V}(p) = \frac{1}{\rho+\delta}\left[\bar{\lambda}\mathbb{I}_{p \geqslant p^*(r_1)} - c + \lambda_\iota(p)\frac{p^2(1-p)^2}{2\sigma^2}\tilde{V}''(p)\right], \quad \tilde{V}(\tilde{\underline{p}}) = \tilde{V}'(\tilde{\underline{p}}) = 0,$$

and let $(V(p), \underline{p})$ solve:

$$V(p) = \frac{1}{\rho+\delta}\left[\bar{\lambda}\mathbb{I}_{p \geqslant p^*(r)} - c + \lambda_\iota(p)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right], \quad V(\underline{p}) = V'(\underline{p}) = 0,$$

Lemma B.9 implies that $\tilde{V}(p) \geqslant V_1(p)$ for all $p \in [0,1]$. But then if $p^*(r) < p^*(r_1)$, it follows from equation (3) that $V(p) > \tilde{V} \geqslant V_\lambda(p)$ for all $p \in [\underline{p}(r_1), 1]$. Evaluating these inequalities at p_0 violates the condition $V(p_0) = K$, thus proving the claim.

B.6 Lemma 7

Fix an equilibrium price path $(q_t)_{t \geqslant 0} \triangleq \{q(p)\}$ and associated value function $V(p)$, and suppose there exists a p such that $\Theta(p, q(p)) < 1$. Let $\Theta(p, q(p)) = 1 - \psi$ for some $\psi > 0$. I claim there exists $q' > q(p)$ such that $(p - q') > (p - q(p))\,\Theta(p, q(p))$, i.e. such a price offers consumers a strictly higher expected value. To see this, let $q' = q + \chi$. Then:

$$\begin{aligned}(p - q') &> (p - q(p))\,\Theta(p, q(p)) \\ p - (q + \chi) &> (p - q(p))\,(1 - \psi) \\ \chi &< (p - q(p))\,\psi,\end{aligned}$$

which holds for sufficiently small χ. (note that $p - q(p) > 0$ in equilibrium) Thus, under the market utility assumption, $\Theta(p, q(p)) < \Theta(p, q') \leqslant 1$ must hold. Finally, note that such a price offer would constitute a profitable deviation for a firm with rating p, since $\Theta(p, q') \leqslant 1$ implies that $\pi(p, q') = \bar{\lambda}$, and hence:

$$\begin{aligned} V(p; q') &= \frac{1}{\rho+\delta}\left[q'\pi(p, q') + (\pi(p, q') + \epsilon)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right] \\ &= \frac{1}{\rho+\delta}\left[q'\bar{\lambda} + (\bar{\lambda} + \epsilon)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right] \\ &> \frac{1}{\rho+\delta}\left[q\bar{\lambda} + (\bar{\lambda} + \epsilon)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right] \\ &= V(p; q) \end{aligned}$$

B.7 Proposition 3

The details of the proof are similar to that of Theorem 1. The outline is as follows:

1. Fix the entry rate η and equilibrium price level w
   (a) Solve explicitly for $(V, \underline{p}, \tilde{p})$
   (b) Solve explicitly for f for arbitrary $\underline{p}, p^*$
2. Prove there exists a unique $w > 0$ such that $V(p_0) = K$
3. Prove there exists a unique $\eta > 0$ such that $p^* = \tilde{p}$

As before, standard arguments yield that the variational inequality in Lemma 6 has a unique value function and associated stopping time and policy. I will guess and verify that the policy takes the threshold form:

$$\pi(p) = \begin{cases} \bar{\lambda} & \text{if } p \geqslant \tilde{p} \\ 0 & \text{if } p < \tilde{p} \end{cases} \tag{B.18}$$

where $\tilde{p}$ satisfies the equation:

$$p - w + \frac{p^2(1-p)^2}{2\sigma^2}V''(p) = 0 \tag{B.19}$$

Under this guess, the value function takes the form:

$$V(p) = \frac{1}{\rho+\delta}\left[\bar{\lambda}(p-a)\pi(p) - c + (\pi(p) + \epsilon)\frac{p^2(1-p)^2}{2\sigma^2}V''(p)\right] \tag{B.20}$$

The general solution to equation (B.20) is as for (5), while the particular solution is now affine. Thus, the full solution is:

$$V(p) = \begin{cases} c_1^0 p^{1-\gamma_1^v}(1-p)^{\gamma_1^v} + \frac{\bar{\lambda}(p-w)-c}{\rho+\delta} & \text{if } p \geqslant \tilde{p} \\ c_0^0 p^{1-\gamma_0^v}(1-p)^{\gamma_0^v} + c_0^1 p^{\gamma_0^v}(1-p)^{1-\gamma_0^v} - \frac{c}{\rho+\delta} & \text{if } p < \tilde{p} \end{cases} \tag{B.21}$$

where the exponents γ_0^v, γ_1^v and coefficients c_0^0, c_0^1, c_1^0 solve the matrix as detailed in Proposition B.1. Simple algebra confirms the verification step (contact author for details). Furthermore, it is readily verified that $V''(p) \geqslant 0$ for all $p \in [\underline{p}, 1]$, and hence that $\tilde{p} \leqslant w$ (clearly $w - w + \frac{p^2(1-p)^2}{2\sigma^2}V''(w) \geqslant 0$). The solutions for the distribution f are identical to the derivations provided in Section B.4.1.

To prove the claim that there exists a unique $w > 0$ such that $V(p_0) = K$, note that the expression B.21 shows that $V(p)$ is continuous and strictly increasing taken as a function of w. As $w \to 0$, $\tilde{p} \to 0$ by the Sandwich Theorem, and hence $V(p_0) \to \frac{p_0\bar{\lambda}-c}{\rho+\delta} > K$, and as $w \to 1$, $V(p_0) \leqslant V(1) \to 0$, thus the claim follows from the Intermediate Value Theorem.

Finally, to prove the claim that there exists a unique $\eta > 0$ such that $\tilde{p} = p^*$, note that $\tilde{p}$ is invariant to η, whereas p^* is continuously increasing in η as proven in Section B.4.3, thus again the claim follows from the Intermediate Value Theorem.

B.8 Proposition 4

The proof employs a similar logic to that of Theorem 2: take the full transparency equilibrium with associated price level w and value function $V_{FT}(p)$. I will construct a private all-or-nothing rating policy $\{\bar{r}\}$ that – taking the price level w as given – would provide strictly greater entry value to firms, i.e. $\bar{V}(p_0) > V_{FT}(p_0)$. Thus, to satisfy the free entry condition, it must be that the equilibrium price level $\bar{w} < w$ under the policy $\{\bar{r}\}$ by the argument in Section B.7. Since $CS = w$ under both policies, this completes the proof.

To wit, suppose under full transparency, firms' value functions are denoted $V_{FT}(p)$, with associated strategies $\underline{p}_{FT}, \tilde{p}_{FT}$ and price level w_{FT}. Consider the private simple all-or-nothing policy $\{\bar{r}\}$ given by $\bar{r}(p) = \mathbb{I}_{p \geqslant \bar{p}}$ for some $\bar{p} \geqslant \max\{p_0, \tilde{p}\}$, with associated equilibrium value function $\bar{V}(p)$, strategies $\bar{\underline{p}}, \tilde{\bar{p}}$ and price level $\bar{w}$. I claim that $\bar{w} < w_{FT}$. Toward a contradiction, assume $\bar{w} = w_{FT}$. I will prove that $\bar{V}(p_0) > V_{FT}(p_0)$, thus violating the free entry condition. Suppose $\bar{V}(p_0) = V_{FT}(p_0)$. Then since $\bar{p} \geqslant \max\{p_0, \tilde{p}\}$ by assumption, V_{FT} and $\bar{V}$ solve the same functional equation on $[0, p_0]$ with shared boundary conditions, and thus it must be that $\tilde{\underline{p}} = \underline{p}_{FT} \triangleq \underline{p}$ and

$\bar{p} = \tilde{\bar{p}}_{FT} \triangleq \tilde{p}$. Using the derivation in Section A.1, we may then write:

$$
\begin{aligned}
V_{FT}(p_0) &= \int_{\tilde{p}}^{1} L(p; p_0, \underline{p}, 1, \rho+\delta) \left[\bar{\lambda}(p-w)\right] dp - c \int_{\underline{p}}^{1} L(p; p_0, \underline{p}, 1, \rho+\delta) dp \\
\bar{V}(p_0) &= \int_{\tilde{p}}^{\bar{p}} L(p; p_0, \underline{p}, 1, \rho+\delta) \left[\bar{\lambda}(p-w)\right] dp + \int_{\bar{p}}^{1} L(p; p_0, \underline{p}, 1, \rho+\delta) \left[\bar{\lambda}\left(\left(\frac{\int_{\bar{p}}^{1} p f(p) dp}{\int_{\bar{p}}^{1} f(p) dp}\right) - w\right)\right] dp \\
&\quad - c \int_{\underline{p}}^{1} L(p; p_0, \underline{p}, 1, \rho+\delta) dp
\end{aligned}
$$

Equating the two expressions then implies that

$$
\begin{aligned}
\frac{\int_{\bar{p}}^{1} p f(p) dp}{\int_{\bar{p}}^{1} f(p) dp} &= \frac{\int_{\bar{p}}^{1} p L(p; p_0, \underline{p}, 1, \rho+\delta) dp}{\int_{\bar{p}}^{1} L(p; p_0, \underline{p}, 1, \rho+\delta) dp} \\
\Leftrightarrow \frac{\int_{\bar{p}}^{1} p^{-\gamma_1^f} (1-p)^{\gamma_1^f - 2} dp}{\int_{\bar{p}}^{1} p^{-1-\gamma_1^f} (1-p)^{\gamma_1^f - 2} dp} &= \frac{\int_{\bar{p}}^{1} p^{-\gamma_1^v} (1-p)^{\gamma_1^v - 2} dp}{\int_{\bar{p}}^{1} p^{-1-\gamma_1^v} (1-p)^{\gamma_1^v - 2} dp}
\end{aligned}
$$

where $\gamma_1^f = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2 \delta}{\epsilon + \bar{\lambda}}}$ and $\gamma_1^v = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\sigma^2(\rho+\delta)}{\epsilon + \bar{\lambda}}}$ are as derived in Section B.4. Thus, a contradiction has been derived since $\rho > 0$. This completes the proof that $V(p_0) > \bar{V}_{FT}(p_0)$, violating free entry and hence completes the proof that $\bar{w} < w_{FT}$. It remains to be shown that equilibrium consumer welfare under $\{r\}$ is equal to $\bar{w}$:

$$
\begin{aligned}
CW &\triangleq \int_{\underline{p}}^{1} (p - q(p))\, \Theta(p) g(p) dp \\
&= \int_{\tilde{p}}^{1} \left[p - (p - \bar{w})\right] g(p) \\
&= \bar{w} \int_{\tilde{p}}^{\bar{p}} f(p) dp + \left[\int_{\bar{p}}^{1} \left[p - \left(\frac{\int_{\bar{p}}^{1} p f(p) dp}{\int_{\bar{p}}^{1} f(p) dp} - \bar{w}\right)\right] f(p) dp\right] \\
&= \bar{w} \int_{\tilde{p}}^{1} f(p) dp \\
&= \bar{w}
\end{aligned}
$$