A class of metrics on the space of probability measures

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Abstract

We introduce a new class of metrics on the set of probability measures $P(X)$ on a domain $X \subseteq \mathbb{R}^n$, based on a refinement of the notion of generalized geodesics with respect to a base measure ν. We characterize these metrics in several ways: as optimal costs among transports which commute with optimal transport to ν, as the integral of optimal costs between each level set of optimal transport to ν, and through limits of certain multi-marginal optimal transport problems. Our class interpolates between the usual quadratic Wasserstein distance (when ν is a Dirac mass) and a metric associated with the uniquely defined generalized geodesics obtained when ν is absolutely continuous with respect to Lebesgue measure. We are especially interested in the cases in between; for instance, when ν concentrates on a lower dimensional submanifold of $\mathbb{R}^n$. For such bases we prove several geodesic convexity results. As applications of these ideas, we prove convergence of an iterative scheme to solve a minimization problem arising in game theory, and characterize solutions to the related multi-marginal optimal transport problem by a differential equation, yielding a potential numerical method for it.

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