Self-assessment guidance

Double degree admission. Telecom Paris & Sciences Po Paris

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The following exercises are examples of questions that might be asked in an interview. The difficulty level is indicated by a star rating. One star means the problem can be solved immediately ; two stars mean the problem can be solved by working through it methodically with a pen and paper ; three stars mean solving the problem requires creativity and exploring different approaches before arriving at a solution.

While presenting a solution to these exercises, expect the examiner to ask questions about the definitions of the objects you are manipulating or about results you are applying (hypothesis of a theorem, extensions, etc.).

1 General reasoning

Exercise 1.1 ($\star$)**.** Let A, B, C, and D be four sets, and let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be three functions. Show that

1. if $g \circ f$ is injective, then f is injective ;
2. if $g \circ f$ is surjective, then g is surjective ;
3. if $g \circ f$ and $h \circ g$ are bijective, then f, g, and h are bijective.

Exercise 1.2 ($\star\star$)**.** Let X and Y be two sets and let $f : X \to Y$ be a function. Recall that 2^X, respectively 2^Y, denotes the power set of X, respectively of Y. We denote by $\hat{f}$ and $\check{f}$ the functions

$$\hat{f} : \begin{cases} 2^X & \to & 2^Y \\ A & \mapsto & f(A) \end{cases} \quad \text{and} \quad \check{f} : \begin{cases} 2^Y & \to & 2^X \\ B & \mapsto & f^{-1}(B) \end{cases}.$$

1. Show that f is injective if and only if $\hat{f}$ is injective.
2. Show that f is surjective if and only if $\check{f}$ is injective.

2 Linear algebra

Exercise 2.1 ($\star$)**.** Let $\mathbb{K}$ be a field. Let $\mathbf{A} \in \mathbb{K}^{m\times n}$ et $\mathbf{b} \in \mathbb{K}^m$. Describe the different possibilities regarding the set of solutions of the system $\mathbf{Ax} = \mathbf{b}$.

Exercise 2.2 ($\star$)**.** Let E be a vector space of finite dimension. Let A and B be two vector subspaces of E.

1. Suppose there exists a vector subspace X of E such that

$$A \oplus X = B \oplus X = E.$$

   (a) What can be said about $\dim A$ in relation to $\dim B$?

   (b) Is the answer to question (1a) still correct if we only assume that

$$A \oplus X = B \oplus X?$$

   Explain why.

   (c) Is the answer to question (1a) still correct if we only assume that

$$A + X = B + X = E?$$

   Explain why.

2. Suppose that $\dim A = \dim B$. Prove that there exists a vector subspace X of E such that

$$A \oplus X = B \oplus X = E.$$

3. In this question, consider the case where $E = \mathbb{R}^4$, where A is the vector space spanned by the vectors

$$\begin{pmatrix}1\\4\\2\\3\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}5\\-3\\1\\8\end{pmatrix}$$

   and where B is the vector space spanned by the vectors

$$\begin{pmatrix}-3\\11\\3\\-2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-7\\2\\4\\-7\end{pmatrix}.$$

   Exhibit a vector subspace X such that

$$A \oplus X = B \oplus X = E.$$

Exercise 2.3 ($\star$)**.** We work in the Euclidean space $\mathbb{R}^3$ equipped with the canonical inner product. Let $\mathbf{x}$ be the vector and (D) the line defined by

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \in \mathbb{R}^3 \quad \text{and} \quad (D) : \begin{cases} x - z & = 0 \\ 3x + 2y - 3z & = 0 \end{cases}$$

Compute the distance between the point $\mathbf{x}$ and the line (D).

Exercise 2.4 ($\star\star$)**.** Let n be an integer and let $a_1, a_2, \dots, a_n$ be scalars in a field$\mathbb{K}$. Compute the $(n+1) \times (n+1)$ determinant

$$D_n(x) = \begin{vmatrix} x & a_1 & a_2 & \dots & a_n \\ a_1 & x & a_2 & \iddots & a_n \\ a_1 & a_2 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_n \\ a_1 & a_2 & \dots & a_n & x \end{vmatrix} \in \mathbb{K}[x].$$

Exercise 2.5 ($\star\star$)**.** Let V be a finite-dimensional vector space. Consider an endomorphism $f : V \to V$. Show that the following three statements are equivalent :

1. $V = \operatorname{im} f \oplus \ker f$;
2. $\operatorname{im} f = \operatorname{im} f^2$;
3. $\ker f = \ker f^2$.

Exercise 2.6 ($\star\star\star$)**.** Let $\mathbb{K}$ be a field. Let $(h_1, h_2, \dots, h_n) \in \mathbb{K}^n$ be n non-zero scalars, and let

$$\mathbf{M} = \begin{pmatrix} 1 + \frac{1}{h_1} & 1 & \cdots & 1 \\ 1 & 1 + \frac{1}{h_2} & 1 & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1 + \frac{1}{h_n} \end{pmatrix} \in \mathbb{K}^{n \times n}$$

Compute the inverse of $\mathbf{M}$ when it exists.

Exercise 2.7 ($\star\star\star$)**.** Let $\mathbb{K}$ be a field. Let $\alpha \in \mathbb{K}$ be a scalar and let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{n \times n}$ be two matrices. Study the solutions for the unknown $\mathbf{X} \in \mathbb{K}^{n \times n}$ of the equation

$$\alpha \mathbf{X} + \operatorname{tr}(\mathbf{X}) \mathbf{A} = \mathbf{B}.$$

3 Calculus

Exercise 3.1 ($\star$). Draw the graph of the function $x \mapsto \max(|x|-1, 0)$, defined on $\mathbb{R}$. Is it convex ? concave ? Does it have a unique argmin ? argmax ?

Exercise 3.2 ($\star\star$). Find the solutions of the following differential equation

$$x(x^2-1)y' + 2y - x^2 = 0.$$

Exercise 3.3 ($\star\star\star$). Show that there is no polynomial $P(X) \in \mathbb{Z}[X]$ such that the value $P(n)$ is a prime number for every integer n. (Hint : Consider $P(n+P(n))$ and Taylor expansion.)

4 Functional analysis

Exercise 4.1 ($\star\star$). Show that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left(1+\frac{t^2}{n}\right)^{-n} \mathrm{d}t = \int_{\mathbb{R}} e^{-t^2}\mathrm{d}t$$

Exercise 4.2 ($\star\star$). Let $f : \mathbb{R} \to \mathbb{R}$ be a L^1 function. Assume that $\int_{\mathbb{R}} |f| = \left|\int_{\mathbb{R}} f\right|$. Show that the function f has a constant sign almost everywhere.

Exercise 4.3 ($\star\star$). Let $f, g : \mathbb{R} \to \mathbb{R}$ be two L^2 function. Show that the product fg belongs to L^1.

Exercise 4.4 ($\star\star$). Let $f : \mathbb{R} \to \mathbb{R}$ be a L^1 function. Assume that f is continuous. Is it the case that $\lim_{x\to\infty} f(x) = 0$?

Exercise 4.5 ($\star\star\star$). Let $]a, b[$ be an open non empty interval of $\mathbb{R}$. Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be two sequences of real numbers. We assume that for any $x \in]a, b[$, the limit $\lim_{n\to+\infty} a_n \sin(nx) + b_n \cos(nx)$ exists and is zero. We would like to prove that the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to 0 when $n \to +\infty$. We use reasoning by contradiction and assume that this is not the case.

1. Show that there exists a strictly increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that, for every $x \in]a, b[$, the sequence of functions

$$f_k(x) = \frac{a_{n_k} \sin(n_k x) + b_{n_k} \cos(n_k x)}{a_{n_k}^2 + b_{n_k}^2}$$

2. Compute in two different ways the limit $\lim_{k\to\infty} \int_a^b f_k(x)\mathrm{d}x$ and find a contradiction.

5 Probability

Exercise 5.1 ($\star$)**.** Let $(\Omega, A, \mathbb{P})$ be a probability space where $\Omega = [0, 1]$, A is the Borel σ-algebra on Ω, and $\mathbb{P}$ is the uniform probability on $[0, 1]$.

— Let X be the random variable such that $X(\omega) = 1 - \omega$ for all $\omega \in \Omega$. Determine the probability distribution of X, its expected value, and its variance.

— Answer the same questions for $Y(\omega) = -\ln(\omega)$.

Exercise 5.2 ($\star$)**.**

1. Let $\mathbf{X} \in \mathbb{R}^p$ be a random vector with covariance $\mathbf{\Sigma} \in \mathbb{R}^p$. Does the matrix $\mathbf{\Sigma}$ always admit a decomposition of the form $\mathbf{UDU}^\top$ where $\mathbf{D} \in \mathbb{R}^{p \times p}$ is diagonal ? If yes, specify the properties and dimensions of $\mathbf{U}$.
2. Suppose that such a decomposition exists and that each diagonal element of $\mathbf{D}$ is strictly positive. Find a matrix $\mathbf{A}$ such that $\mathrm{cov}(\mathbf{AX}) = \mathbf{I}_p$ where $\mathbf{I}_p$ is the identity matrix of size $p \times p$.

Exercise 5.3 ($\star$)**.** Give the density of a Gaussian vector with mean $\mu \in \mathbb{R}^d$ and variance $\mathbf{\Sigma} \in \mathbb{R}^{d \times d}$.

Exercise 5.4 ($\star\star$)**.** Let $(Y_i)_{i \in \mathbb{N}_{>0}}$ be a sequence of independent Bernoulli random variables with common parameter p. What is the support of the random variable $Z_n = \sum_{i=1}^n y_i$? What is the probability that $Z = k$? What is the name of the distribution Z ?

Exercise 5.5 ($\star\star$)**.** Let $\mathbf{A} \in \mathbb{R}^{p \times p}$, $\mathbf{b} \in \mathbb{R}^p$ and $\mathbf{X}$ be a random vector valued in $\mathbb{R}^p$ with covariance $\mathbf{\Sigma}$. Express $\mathrm{cov}(\mathbf{AX})$ in terms of $\mathbf{\Sigma}$, $\mathbf{A}$ and $\mathbf{b}$.

Exercise 5.6 ($\star\star\star$)**.** Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. (independent and identically distributed) such that $m_2 = \mathbb{E}(Y_1^2) < \infty$ and $\mathbb{E}[Y_1] = 0$. Let $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n Y_i$. Express $\mathbb{E}[\hat{\mu}^2]$ using m_2 and n.

Exercise 5.7 ($\star\star\star$)**.** Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. (independent and identically distributed) such that $\mathbb{E}(Y_1^2) < \infty$. We want to estimate $\mu_0 = \mathbb{E}[Y_1]$. Let $(w_i)_{1 \le i \le n}$ be strictly positive and deterministic weights. Give $\hat{\mu}$ which minimizes $\sum_{i=1}^n w_i (Y_i - \mu)^2 + \lambda \mu^2$ where $\lambda \ge 0$? Express $\mathbb{E}[\hat{\mu}]$.

6 Statistics

Exercise 6.1 ($\star\star$)**.** Let $(X_1, \ldots, X_n)$ be an n-sample of i.i.d. random variables following a Poisson distribution $P(\theta)$, where $\theta > 0$ is unknown. Determine a maximum likelihood estimator for θ. Is it biased ? Does it converge ? Is it efficient ?

Exercise 6.2 ($\star\star$)**.** Ten 1,500-meter runners are testing whether taking a non-doping dietary supplement improves their performance. They run a 1,500-meter race without the product (or rather, with a placebo) and, two days later, another 1,500-meter race with the product. Let $D_1, \ldots, D_{10}$ denote the difference between their race times without and with the dietary supplement. We assume that the $(D_i)_{1\leq i\leq 10}$ are valid Gaussian random variables. We want to test the hypothesis (H_0) : the product has no effect, against the hypothesis (H_1) : the product has a positive effect.

Among the 10 runners, we found an average time difference of 3 seconds between the two races, with a standard deviation of 4 seconds. Determine the result of the test with a 5% significance level.

7 Optimisation

Exercise 7.1 ($\star$)**.** Compute the differential of the following $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ applications (at any point) :

1. $\mathbf{M} \mapsto \mathbf{M}^\top \mathbf{M}$.
2. $\mathbf{M} \mapsto \mathbf{M}^3$.

Exercise 7.2 ($\star\star$)**.** Let $(a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Give the gradient of the function $\mathbf{x} \mapsto \left(y - \sum_{k=1}^d a_k x_k\right)^2$ defined on $\mathbb{R}^d$. Does this function have a unique minimum ?

Exercise 7.3 ($\star\star$)**.** Let $T \subseteq \mathbb{R}^2$ be the set

$$T = \left\{(x, y) \in \mathbb{R}^2 \ : x \geq 0,\ y \geq 0,\ x + y \leq 1\right\}$$

Let f be the function given by

$$f(x, y) = -x - 2y - 2xy + 1 + \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

1. Is the function f convex ? Is it concave ?
2. Prove that f indeed has a minimum and a maximum on T.
3. Prove that every local minimum or maximum of f on T lies on the boundary of T.
4. Find the minimum and maximum of f on T.