Directed search with phantom vacancies

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Abstract

When vacancies are filled, the ads that were posted are often not withdrawn. Stale information about already filled vacancies implies that old ads are more likely to be obsolete than are younger ones. When ads for jobs stipulate the vacancy creation date, job-seekers apply for the different jobs so as to equalize matching odds across vacancy age. This search behavior leads them to over-apply for young ads. Thus filling a vacancy of a given age creates a negative informational externality that affects all cohorts of vacancies after this age. The magnitude of the externality decreases with the age of the filled vacancy. We calibrate the model using US labor market data. The contribution of phantom vacancies to overall frictions is large, although the magnitude of the externality is small.

JEL codes:

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1 Introduction

I am currently on the job hunt and I had a question about applying to jobs online. You know how most websites will tell you the job has been posted 1 day ago, 28 days ago, etc. For some reason, I have concluded that I need to apply to a job the first week they post the position to have the best chances of being hired. Although I heard that it can take up to a month for the company to hire anyone for the position, I feel that applying to a job that was posted 3 weeks ago isn’t that promising. What is your take on this situation?

AskManager.com

This quote illustrates two important points about the labor market. First, some of the information available to job seekers is out of date. Some advertised jobs have already been filled. These ads are for job openings that no longer exist; they are ads for phantom vacancies. Second, job seekers know that some advertised jobs are phantoms, and they adjust their application behavior accordingly. As jobs advertised in older listings are more likely to be phantoms, workers may decide to apply to more recently posted vacancies, but at the same time responding to older ads is also an option since fewer applicants are likely to be pursuing these jobs. Workers thus face a problem of directed search, namely, how to allocate their applications optimally across vacancy listings of different ages.

In this paper, we explore the equilibrium and efficiency implications of directed search with phantoms. In equilibrium, worker directed search satisfies an indifference condition – the expected payoffs associated with applying to ads of different ages must be equalized. However, even though workers direct their search optimally, equilibrium is not constrained efficient. From the collective perspective, workers over-apply to young vacancy postings. When a worker takes a job, a phantom is added to the market, and the cost of the phantom is greater the younger was the ad that was posted for the job. That is, the social cost of a young phantom is greater than that of an old phantom. To see this, consider a match formed with a vacancy that was posted one month ago. This generates a one-month old phantom. This phantom can only affect the job seekers who pursue listings that are older than one month. However, a match formed with a newly posted vacancy generates a phantom that can affect all job seekers during its lifetime.

Our paper is related to four strands of literature. First, the paper closest to ours is Chéron and Decreuse (2014). Their paper introduces the concept of phantom vacancies and shows how phantoms lead to an aggregate matching function even in the absence of other coordination frictions in the market. The authors then embed this matching function in the Diamond-Mortensen-Pissarides equilibrium search model and examine how the dynamics associated with
phantom creation and destruction affect the business cycle properties of that model. One thing they do not do, however, is allow workers to direct their search. Our paper is thus a natural complement to Chéron and Decreuse (2014). Second, there is a literature on inefficiencies in search equilibrium that come from composition externalities, e.g., Albrecht, Navarro and Vroman (2010) and Chéron, Hairault and Langot (2011). These are models with worker heterogeneity in which individual decisions (to participate in the labor market, to form or dissolve matches) affect the distribution of worker types across the pool of unemployment. A related compositional effect obtains in our model. A worker who successfully targets a job listing of a particular age changes the distributions by age of both vacancies and phantoms but does not take the effect of these changes on other job seekers into account. Third, our paper is related to the literature on stock-flow matching, originating with Coles and Smith (1998), in which job seekers initially search through the entire stock of vacancies but thereafter only look at the flow of new vacancies. The connection is that both our model and the stock-flow model predict that younger vacancies have a better chance of being filled quickly than do older vacancies, albeit for different reasons. Finally, there is a substantial empirical literature based on data culled from job search engines. Examples include Faberman and Kudlyak (2014) using data from SnagAJob and Marinescu (2015) using data from CareerBuilder.

The outline of the rest of the paper is as follows. In the next two sections, we lay out our model of directed search with phantoms. After computing the allocation generated by random search as a benchmark, we characterize the directed search allocation in which job seekers decide where to apply based on the age of job listings. We derive the steady-state unemployment rate for both allocations as well as the distributions by age of phantoms and of vacancies. Since the two allocations are described by how job seekers distribute their applications by age, we can also derive the distribution of job seekers across job listings of different ages. We then describe the constrained efficient allocation, that is, the allocation of job seekers across ads of different ages that a social planner would choose, and prove that neither the random search nor the directed search allocation is constrained efficient. Although it is not possible to completely solve for the efficient allocation, we describe an “almost efficient” allocation that helps us better understand the inefficiencies associated with the random search and directed search allocations.

In Section 4, we provide a quantitative example based on a calibration for the U.S. labor market. In this example, eliminating phantoms from the labor market would cut the unemployment rate by one third, and two thirds of overall frictions can be attributed to information obsolescence. However, the difference in unemployment rates between the worst allocation (random search) and the almost efficient allocation is only 0.5 percentage points, and quantitatively the directed search allocation does not differ much from the almost efficient allocation. The
reason that phantoms are quantitatively so important even in the directed search and almost efficient allocations is that efficient search methods generate more matches, thus creating more phantoms.

Finally, Section 5 is devoted to three extensions. We first examine what happens when job listings are renewed at random intervals of time. Vacancy renewal means that a listing for an unfilled vacancy is replaced by a new one. This practice concentrates the distribution of vacancies in young ages. Job seekers react by applying to younger listings as compared with the allocation without renewal. The qualitative conclusion stays the same: directed search is not constrained efficient. Quantitatively, the spread between the random search allocation and the almost efficient allocation grows to 1.2 percentage points. At the same time, the contribution of phantoms to overall frictions decreases to 40%. Second, we discuss the role of prices in the allocation of job seekers across market segments. Fixed-wage contracts cannot internalize the vacancy-age-dependent informational externality. One could imagine firms posting sophisticated contracts advertising a wage that varies with the length of time it takes to fill the vacancy, but contracts of this sort are not realistic. Instead, and as an approximation to these more sophisticated contracts, we consider Nash bargaining over the wage. With Nash bargaining, the wage increases with vacancy age. The reason is that the value a vacancy falls with age because workers are less likely to apply for older vacancies. Thus the match surplus increases with vacancy age, and the wage does the same. Workers then have a stronger incentive to apply for older jobs even though the job-finding rate falls with age. Finally, we consider a complementary reason for why job-seekers pay attention to the vacancy creation date. Some jobs are lemons, i.e., jobs that no one can or wants to accept. Because no one accepts these jobs, they stay in the distribution of job postings until they naturally disappear. The proportion of job listings that are lemons increases with age, and this composition effect pushes job seekers to direct their search towards younger listings. Lemons generate an additional inefficiency: job seekers who direct their search towards younger listings increase the proportion of lemons in the stock of job listings. However, a key difference between lemons and phantoms is that matching does not produce lemons. As a result, the quantitative contribution of lemons to overall unemployment is small unless the fraction of lemons in new listings is very large.

2 The model

We focus on the stationary state of a continuous time model.

Calendar time is denoted by $t$. At each time, there is a fixed measure of jobs $K$ that can be either vacant or occupied. The measure of vacancies is $v(t)$. They differ in age $a \geq 0$. There
is also a continuum of workers of total size one. Each worker can be either unemployed or employed. The measure of unemployed is $u(t)$. By construction, we have $v(t) + 1 - u(t) = K$.

Employed workers and jobs separate at exogeneous Poisson rate $\lambda$. Newly unemployed workers join the pool of unemployed and immediately start job search. Newly destroyed jobs join the pool of vacancies.

All jobs produce the flow output $y \equiv 1$ and pay the same wage $w$. The search market is segmented by vacancy age $a$. In each submarket $a$, $u(a, t)$ unemployed workers try to match with $v(a, t)$ vacancies. The matching process is frictional. On top of usual search frictions, information persistence about former vacancies creates an additional source of matching frictions. Each time a match is formed and the corresponding ad is not withdrawn, a phantom is created. The flow of new matches is

$$M(a, t) = \frac{v(a, t)}{v(a, t) + p(a, t)} m(u(a, t), v(a, t) + p(a, t)), \quad (1)$$

where $p(a, t)$ is the measure of phantom vacancies.

The function $m$ is strictly concave, has constant returns to scale, and is such that $m(0, v) = m(u, 0) = 0$, $\lim_{u \to 0} m_1(u, v + p) = \lim_{v \to p} m_2(u, v + p) = \infty$. The flow of new matches has two components: the contact rate $m$ multiplied by the proportion of contacts where a vacancy rather than a phantom is involved. Phantoms deteriorate the search process: by increasing $p$ at given $v$ and $u$, we decrease $M$.

We define $\theta(a, t) \equiv (v(a, t) + p(a, t))/u(a, t)$ as the tightness in the submarket corresponding to ads of age $a$. The numerator is composed of the total measure of age-$a$ ads, thus including phantom and nonphantom vacancies. We also define $\pi(a, t) \equiv v(a, t)/[v(a, t) + p(a, t)]$ as the nonphantom proportion.

Unemployed workers spread themselves over the different submarkets with total unemployment at time $t$ of $u(t) = \int_0^\infty u(a, t)da$. Unemployment obeys the law of motion:

$$du(t)/dt = - \int_0^\infty M(a, t)da + \lambda(1 - u). \quad (2)$$

The stocks of vacancies and phantoms evolve according to the following laws of motion:

$$\partial v(a, t)/\partial a + \partial v(a, t)/\partial t = -M(a, t), \quad (3)$$

$$\partial p(a, t)/\partial a + \partial p(a, t)/\partial t = \beta M(a, t) - \delta p(a, t), \quad (4)$$

with $\beta, \delta \geq 0$, $v(0, t) = \lambda(1 - u(t))$, $p(0, t) = 0$, and $v(t) = \int_0^\infty v(a, t)da$. The measure of vacancies decreases across age by the measure of new matches. A fraction $\beta$ of matches create phantoms, while phantoms depreciate at rate $\delta$. 

5
The law of motion for vacancies implies that vacancies cannot be renewed or refreshed. That is, a job that became available at date \( t \) and which is still available at date \( t + a \) is identified as having age \( a \), and nothing can be done to signal that this vacancy is still active rather than a phantom. Section 4 analyzes the case where vacancies can be renewed, i.e., relisted as new vacancies.

The nonphantom proportion of ads, \( \pi(a, t) \), evolves according to

\[
\frac{\partial \pi(a, t)}{\partial a} + \frac{\partial \pi(a, t)}{\partial t} = -\frac{M(a, t)}{v(a, t)} \pi(a, t) \left[ 1 - (1 - \beta) \pi(a, t) \right] + \delta \pi(a, t) (1 - \pi(a, t)),
\]

with \( \pi(0, t) = 1 \).

In steady state, calendar time does not affect any of the variables. Thus \( du/\partial t = dv/\partial t = 0 \), \( \partial v(a, t)/\partial t = \partial p(a, t)/\partial t = 0 \), \( u(a, t) = u(a) \), and \( M(a, t) = M(a) \). Hereafter we refer to variables without mentioning calendar time again. A dot over a variable \( x \) denotes the derivative of \( x \) with respect to age \( a \), i.e. \( \dot{x} \equiv x'(a) \).

For later use, we define the elasticity of the meeting function with respect to tightness as \( \alpha(\theta) = \theta m_2(1, \theta)/m(1, \theta) \), the job-finding rate by vacancy age as \( \mu(a) = M(a)/u(a) \), and the vacancy-filling rate by vacancy age as \( \eta(a) = M(a)/v(a) \). Finally, \( \phi_u \), \( \phi_v \), and \( \phi_j \) denote the density functions of unemployment, vacancies and the total measure of ads (vacancies + phantoms) by age. By definition, we have \( \phi_u(a) = u(a)/u \), \( \phi_v(a) = v(a)/v \), and \( \phi_j(a) = (v(a) + p(a))/(v + p) \) for all \( a \geq 0 \).

To close the model, we need to define how the job-seekers spread over the different submarkets.

3 Random search, directed search, and constrained-efficient allocations

We describe three different allocations. Each one is associated to a particular distribution of job-seekers over vacancy ages.

3.1 Random search allocation

Before we derive the directed search allocation, we start, as a baseline, with the allocation obtained when workers do not observe vacancy age, or, equivalently, when they cannot take this information into account. Thus search is random and the ratio of vacancies and phantoms to job-seekers is constant over age.

For all \( a \geq 0 \),

\[
\theta(a) = (v(a) + p(a))/u(a) = \theta.
\]

6
Vacancies and phantoms evolve according to:

\[ \dot{v} = -v(a) \frac{m(1, \theta)}{\theta}, \tag{7} \]
\[ \dot{p} = \beta v(a) \frac{m(1, \theta)}{\theta} - \delta p(a), \tag{8} \]

with \( v(0) = v_0 = \lambda(1 - u) \) and \( p(0) = 0 \). This gives two ordinary differential equations in \( v \) and \( p \).

The solution is

\[ v(a) = v_0 \exp(-a m(1, \theta)/\theta), \tag{9} \]
\[ p(a) = \sigma v_0 \left[ \exp(-a) - \exp(-a m(1, \theta)/\theta) \right], \tag{10} \]

with \( \sigma = \beta(m(1, \theta)/\theta)/(m(1, \theta)/\theta - \delta) \).

Using the random search property (6), the unemployment rate by vacancy age follows

\[ u(a) = \frac{v_0}{\theta} \left[ \sigma \exp(-a) + (1 - \sigma) \exp(-a m(1, \theta)/\theta) \right]. \tag{11} \]

The overall unemployment rate is \( u = \int_0^\infty u(a) da \). Using \( v_0 = \lambda(1 - u) \), we obtain

\[ u = \frac{\lambda + \lambda(\beta/\delta)m(1, \theta)/\theta}{m(1, \theta) + \lambda + \lambda(\beta/\delta)m(1, \theta)/\theta}. \tag{12} \]

Equation (12) is the Beveridge curve. It highlights the role played by phantoms. Holding tightness constant, unemployment increases with the ratio \( \beta/\delta \). Phantoms create a negative informational externality that reduces the efficiency of the matching process. As this externality is entirely due to job creation, its magnitude increases with the rate at which vacancies are filled \( m(1, \theta)/\theta \). Thus it decreases with tightness.

To close the model, we use the resource constraint \( 1 - u + v = K \).

**Proposition 1** The random search allocation is characterized by the unique \( \theta_{rs} \) that solves

\[ \frac{m(1, \theta_{rs}) + \lambda \theta_{rs}}{m(1, \theta_{rs}) + \lambda \frac{\beta}{\delta} m(1, \theta_{rs})/\theta} = K. \tag{13} \]

The proof is given in Appendix A.

There is a unique random search allocation. In this allocation, tightness increases with the phantom birth rate to death rate ratio, i.e., \( d\theta_{rs}/d(\beta/\delta) > 0 \). This ratio governs the quantitative impact of the negative informational externality that affects the matching process. When the ratio increases, the phantom stock increases as well, all else equal. Thus the ratio of vacancies plus phantoms to unemployment goes up.
The corollary is that unemployment increases with $\beta/\delta$. To see this, we write the resource constraint as:

$$u = 1 - \frac{K}{(1 + \lambda \theta_{rs}/m(1, \theta_{rs}))}.$$ 

This equation is particularly convenient because the ratio $\beta/\delta$ only affects the right-hand side through changes in $\theta_{rs}$. An increase in $\beta/\delta$ implies that $m(1, \theta_{rs})/\theta_{rs}$ goes down and so unemployment must rise.

In the random search allocation, the rate at which vacancies are filled is $\eta(a) = m(1, \theta_{rs})/\theta_{rs}$ for all $a \geq 0$. It is constant over vacancy age. Thus the chance of being filled does not change with vacancy age. However, the job-finding rate varies with age. The job-finding rate is $\mu(a) = m(1, \theta_{rs})\pi(a)$. The proportion $\pi(a)$ decreases over age, and so the job-finding rate goes down.

Formally, the motion of the nonphantom proportion is

$$\dot{\pi} = -\pi(a)[m(1, \theta_{rs})/\theta_{rs} - \delta - \pi(a)((1 - \beta)m(1, \theta_{rs})/\theta_{rs} - \delta)].$$

We have $\dot{\pi}(0) < 1$. Then two cases are possible. If $m(1, \theta_{rs})/\theta_{rs} \geq \delta$, then vacancies decrease at a faster rate than phantoms do, and $\pi(a)$ monotonically converges towards 0. It does not reach 0 in finite age. Thus the exit rate $\mu(a)$ strictly decreases from $m(1, \theta_{rs})$ to 0. If, on the contrary, $m(1, \theta_{rs})/\theta_{rs} < \delta$, then $\pi(a)$ monotonically converges towards the threshold value $\pi_{rs} = (\delta - m(1, \theta_{rs})/\theta_{rs})/ (\delta - (1 - \beta)m(1, \theta_{rs})/\theta_{rs})$. This value is reached in a finite age. Thus the exit rate strictly decreases from $m(1, \theta_{rs})$ to $m(1, \theta_{rs})\pi_{rs}$, and stays constant afterwards.

The density functions of vacancies, total ads, and job-seekers by vacancy age are

$$\phi_v(a) = \exp\left(-am(1, \theta_{rs})/\theta_{rs}\right)/m(1, \theta_{rs})/\theta_{rs},$$

$$\phi_u(a) = \phi_j(a) = \frac{(1 - \sigma_{rs})\exp(-am(1, \theta_{rs})/\theta_{rs}) + \sigma_{rs}\exp(-\delta a)}{(1 - \sigma_{rs})m(1, \theta_{rs})/\theta_{rs} + \sigma_{rs}/\delta}.$$ 

Vacancies are distributed according to an exponential law. This distribution obtains in standard matching models. Thus it also holds when there are phantoms and search is random. The density functions of total ads and of job-seekers coincide. This is a direct implication of the random search assumption whereby the ratio of total ads to job-seekers does not change across vacancy age. These density functions are composed of two exponential terms. Since $\sigma_{rs}$ is not necessarily positive, these terms can evolve in opposite directions.

When there are no phantoms, which happens when $\beta = 0$ or $\delta$ tends to infinity, unemployment is minimized. In this case, both $\eta(a)$ and $\mu(a)$ are constant over vacancy age, and all distributions follow the same exponential law. Phantoms increase unemployment and distort the distribution of vacancies and unemployed towards older vacancy ages.
3.2 Directed search allocation

Workers now observe vacancy age and choose which market to enter. A sub-market \( a \) is open if and only if \( u(a) > 0 \). We denote by \( \Omega \) the set of open sub-markets. All open sub-markets must yield the same utility. Thus the job-finding rate \( \mu(a) = M(a)/u(a) \) must be the same across open sub-markets. Given that \( v(a) > 0 \) for all \( a \geq 0 \), the properties of the meeting function \( m \) imply that \( u(a) > 0 \). If not, a worker who would accept to enter a non-open sub-market would immediately find a job. Thus all sub-markets are open in equilibrium and so \( \Omega = \mathbb{R}_+ \).

That \( \mu(a) \) is constant across age implies for all \( a \geq 0 \) that \( M(a)/u(a) = m(1, \theta(0)) \). Using the definition of \( M \) and the definition of \( \pi(a) \), we obtain

\[
m(1, \theta(a))\pi(a) = m(1, \theta(0)). \tag{15}
\]

This equation defines a strictly decreasing relationship between \( \theta(a) \) and \( \pi(a) \). The relationship is parameterized by \( \theta(0) \). Unlike the random search allocation, tightness is not constant in the directed search allocation. It strictly increases with \( \theta(0) \) and strictly decreases with \( \pi(a) \).

Differentiating (15) with respect to age gives:

\[
-\alpha(\theta) \frac{\dot{\theta}}{\theta(a)} = \frac{\dot{\pi}}{\pi(a)}. \tag{16}
\]

Using (15) and (16), we can characterize the motions of these variables as follows:

\[
\dot{\pi} = -\frac{m(1, \theta(0))}{f(\pi(a), \theta(0))} ((1 - \pi(a)) + \beta \pi(a)) + \delta \pi(a)(1 - \pi(a)), \tag{17}
\]

\[
-\alpha(\theta) \dot{\theta} = \left[ 1 - \frac{m(1, \theta(0))}{m(1, \theta(a))} \right] (\delta \theta - m(1, \theta(a)) - \beta m(1, \theta(0))), \tag{18}
\]

where \( f(\pi(a), \theta(0)) \) is defined by \( m(1, f(\pi(a), \theta(0)))\pi(a) = m(1, \theta(0)) \). Both differential equations can be solved given \( \pi(0) = 1 \) and \( \theta(0) \). Note, however, that this latter value can only be found once the equilibrium is solved.

The stationary number of unemployed balances inflows and outflows. That the exit rate from unemployment does not vary with vacancy age considerably simplifies the calculation. Indeed, outflows are \( \int_0^\infty M(a)da = m(1, \theta(0))u \). Thus

\[
u = \frac{\lambda}{\lambda + m(1, \theta(0))}. \tag{19}\]

The number of vacancies by age follows

\[
\dot{v} = -\frac{m(1, \theta(a))}{\theta(a)} v(a)
\]

Therefore

\[
v(a) = v(0) \exp \left[ -\int_0^a \frac{m(1, \theta(b))}{\theta(b)} db \right]. \tag{20}\]
The resource constraint finally writes

\[ 1 - u + \int_0^\infty v(a)da = K. \] (21)

**Proposition 2** The directed search allocation is characterized by the function \(\theta_{ds}(a)\) that solves

\[ -\alpha(\theta_{ds})\theta_{ds} = \left[ 1 - \frac{m(1, \theta_{ds}(0))}{m(1, \theta_{ds}(a))} \right] [\delta \theta_{ds} - m(1, \theta_{ds}(a)) - \beta m(1, \theta_{ds}(0))], \] (22)

\[ \frac{m(1, \theta_{ds}(0))}{\lambda + m(1, \theta_{ds}(0))} \left\{ 1 + \lambda \int_0^\infty \exp \left[ - \int_0^a \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right] da \right\} = K. \] (23)

The proof is given in Appendix A.

The differential equation (22) describes tightness by vacancy age at given initial condition \(\theta(0)\). The resource constraint (23) then determines \(\theta(0)\).

The rate at which vacancies are filled is \(\eta(a) = m(1, \theta_{ds}(0))/(\pi(a)\theta_{ds}(a))\). It decreases over vacancy age. The job-finding rate is \(\mu(a) = m(1, \theta_{ds}(0))\). It is constant over vacancy age. This property naturally arises in a context where there are no phantoms. Then, the ratio of vacancies to unemployed stays constant, and the resulting job-finding rate must be the same at all vacancy ages. Accounting for phantoms modifies this scenario though the result is the same. Workers who take into account obsolete information apply to younger vacancies. Thus tightness increases over vacancy age and the hazard rate of filling vacancies decreases with age.

The density functions of nonphantom vacancies, total vacancies, and job-seekers by vacancy age are

\[ \phi_v(a) = \frac{\exp \left[ - \int_0^a \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right]}{\int_0^\infty \exp \left[ - \int_0^c \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right] dc}, \]

\[ \phi_j(a) = \frac{m(1, \theta_{ds}(a)) \exp \left[ - \int_0^a \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right]}{\int_0^\infty m(1, \theta_{ds}(c)) \exp \left[ - \int_0^c \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right] dc}, \]

\[ \phi_u(a) = \frac{(m(1, \theta_{ds}(a))/\theta_{ds}(a)) \exp \left[ - \int_0^a \frac{m(1, \theta_{ds}(b))}{\theta_{ds}(b)} db \right]}{\int_0^\infty (m(1, \theta_{ds}(c))/\theta_{ds}(c)) \exp \left[ - \int_0^c m(1, \theta_{ds}(b))/\theta_{ds}(b) db \right] dc}. \]

Unlike the random search allocation, the density of job-seekers by vacancy age \(\phi_u\) differs from the density of overall jobs \(\phi_j\). The former strictly decreases over age, whereas the latter may increase. When there are no phantoms, the directed search allocation coincides with the random search allocation. Thus exit rates are constant across vacancy ages, and the distributions of job-seekers and vacancies by age follow exponential laws.

### 3.3 Constrained-efficient allocation

The planner observes vacancy age and decides on the allocation of unemployed people to the different sub-markets. To justify our focus on steady-state allocations, we assume the discount rate is equal to zero. Thus the efficient allocation minimizes the unemployment rate.
The efficient allocation solves the following optimal control problem:

\[
\max_{u(.)} \int_0^\infty u(a)da
\]

subject to

\[
\begin{align*}
\dot{v} &= -m(\pi(a)u(a), v(a)), \\
v(0) &= \lambda \left(1 - \int_0^\infty u(a)da\right), \\
\dot{p} &= \beta m(\pi(a)u(a), v(a)) - \delta p(a), \\
p(0) &= 0, \\
\lambda \left(1 - \int_0^\infty u(a)da\right) &= \int_0^\infty m(\pi(a)u(a), v(a))da, \\
1 - \int_0^\infty u(a)da + \int_0^\infty v(a)da &= K.
\end{align*}
\]

The planner is subject to the evolution of vacancies over age \((c1)-(c2)\), the evolution of phantoms over age \((c3)-(c4)\), the inflow-outflow constraint \((c5)\), and the resource constraint \((c6)\).

**Proposition 3** In the efficient allocation tightness has the following form:

\[
\begin{align*}
\left[\alpha(\theta) - \frac{\theta \alpha'(\theta)}{1 - \alpha(\theta)}\right] \dot{\theta} &= -\pi m(1, \theta) [\alpha(\theta) + b_0 (1 - \alpha(\theta)) \theta] \\
&- \delta (1 - \pi) \theta + \beta \pi m(1, \theta) [\alpha(\theta) + \delta y(1 - \alpha(\theta)) \theta],
\end{align*}
\]

where \(b_0\) is a constant term and the function \(y : [0, \infty) \rightarrow (-\infty, \infty)\) is such that

\[
y(a) = e^{-\delta a} \left[ y_0 + \int_0^a e^{\delta r} \theta(r)^{-1} dr \right].
\]

The proof is given in Appendix A.

Proposition 3 only gives the shape of the efficient allocation. Formally, the efficient tightness function depends on two parameters, \(b_0\) and \(y_0\), which still need to be found. However, having the shape of the solution allows us to eliminate potential candidates.

**Proposition 4** The following statements hold:

(i) The random search allocation differs from the efficient allocation if and only if \(\beta > 0\);

(ii) The directed search allocation generically differs from the efficient allocation when \(\beta > 0\).
The proof is given in Appendix A.

We prove Proposition 4 by contradiction. We assume in turn that each of the proposed allocation is efficient, and we reach a formal impossibility. For the random search allocation, we show that \( \theta(a) = \theta_0 \) for all \( a \geq 0 \) cannot solve equations (24) and (25). Thus randomizing over the different vacancies without accounting for their age is not efficient. This result is not surprising. Still, it differs from standard search models where applying for the different jobs with equal probability is actually efficient. The reason here is of course that older vacancies have a higher probability of being phantom vacancies.

For the directed search allocation, we suppose that there are two functions \( \theta(a) \) and \( y(a) \) that solve equations (22) on the one hand and (24) and (25) on the other hand. We show that this generally requires that \( \theta \) takes a fixed number of discrete values, which violates the fact that \( \theta \) takes a continuum of values in the directed search allocation. Thus, and this is more surprising than in the random search case, the directed search allocation is also inefficient. Equalizing payoffs across vacancy ages conveys a negative externality. We explain it below with a parameterized version of the model.

The efficient allocation is not easy to find. To overcome computational issues and understand the nature of inefficiency better, we consider a range of allocations parameterized by \( \varepsilon \in [0, 1] \):

\[
a \hat{\theta}_\varepsilon = \varepsilon \left\{ -(1 - \beta)m(1, \theta_{\varepsilon 0}) + m(1, \theta_\varepsilon) - \delta \theta_\varepsilon + \delta \theta_\varepsilon m(1, \theta_0)/m(1, \theta_\varepsilon) \right\}.
\]

For a given \( \varepsilon \), we can solve this equation together with the resource constraint \( 1 - u + v = K \) to compute the unemployment rate \( u_\varepsilon \), the job-finding rate by vacancy age \( \mu_\varepsilon(a) \), and the different distributions \( \phi^\varepsilon_u, \phi^\varepsilon_v, \) and \( \phi^\varepsilon_j \). When \( \varepsilon = 0 \), \( \theta_\varepsilon(a) = \theta_{rs} \) for all \( a \geq 0 \). When \( \varepsilon = 1 \), \( \theta_\varepsilon(a) = \theta_{ds}(a) \) for all \( a \geq 0 \). As \( \varepsilon \) goes from 0 to 1, we obtain allocations that go from the random search to the directed search allocations.

This set of allocations parameterized by \( \varepsilon \) has two main advantages. First, we can restrict our quest of the best allocation to this set. By continuity, \( u_\varepsilon \) admits a minimum on the intervalle \([0, 1]\). We call the corresponding allocation the almost efficient one. The set strongly constrains tightness by vacancy age and thus is unlikely to contain the efficient allocation. However, the almost efficient allocation is much easier to find than the efficient one. Second, we already know that \( \theta_{ds} \) strictly increases over age, whereas \( \theta_{rs} \) is flat. By changing the value of \( \varepsilon \), we modify the slope of \( \theta_\varepsilon \). More precisely, the slope increases with \( \varepsilon \), whereas \( \theta_0 \) decreases with \( \varepsilon \). As we shall see in the parameterized case below, the almost efficient allocation implies \( \varepsilon < 1 \). Thus the tightness-vacancy age profile is steeper in the directed search allocation than in the almost efficient one. This provides a natural explanation to the constrained inefficiency of the directed search allocation.
4 Calibration

We suppose that the meeting technology is Cobb-Douglas, i.e. \( m(u, v + p) = m_0 u^{1-\alpha} (v + p)^\alpha \), \( \alpha \in (0,1) \). As usual, \( m_0 \) is the scale parameter that governs total factor productivity. The elasticity \( \alpha \) belongs to \( (0,1) \). We use JOLTS and BLS data for the period 2000-2008. Over this period, the monthly probability of finding a job was about \( \mu_m = 0.4 \). Thus \( 1 - \exp(-\mu) = 0.4 \) and \( \mu = -\ln(1 - 0.4) \approx 0.5 \). The monthly job loss probability was \( \lambda = 0.03 \). This gives \( \mu = -\ln(1 - 0.03) = 0.03 \). In the JOLTS dataset, firms declare how many vacancies they have. Thus the dataset provides a measure of \( v \). The mean ratio of vacancies to unemployed was about \( x = v/u = 0.5 \). As for the phantom parameters, we assume that each match gives birth to a new phantom and so \( \beta = 1.0 \). We also assume that phantoms reach one month with probability 0.5. Thus \( \delta_m = 0.5 \) and so \( \delta = -\ln(1 - 0.5) \approx 0.7 \), i.e. phantoms live for 1.4 months on average. As for \( \alpha \), the aggregate matching technology is \( M = m_0 \int_0^\infty \pi(a)u(a)\theta(a)^\alpha da \). The elasticity of this function with respect to the overall tightness depends on the nonphantom proportion at each age and on the allocation of job-seekers across age. Intuitively, this elasticity should be larger than \( \alpha \): having more vacancies reduces the nonphantom proportion, thereby increasing the number of matches. The value used by Shimer (2005) is 0.34. So we choose \( \alpha = 0.2 \). The final parameter is \( m_0 \). We set it so that the mean job-finding rate of the directed search allocation is roughly equal to \( \mu \). It turns out that \( m_0 = 1.0 \) gives a very close value and so we set \( m_0 = 1.0 \).

Table 1 gives the set of parameters.

<table>
<thead>
<tr>
<th>( m_0 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( \lambda )</th>
<th>( K )</th>
<th>( \mu_{ds} )</th>
<th>( u_{ds} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.2</td>
<td>1.0</td>
<td>0.7</td>
<td>0.03</td>
<td>0.972</td>
<td>0.487</td>
<td>0.059</td>
</tr>
</tbody>
</table>

Table 1: Parameter values

Figure 1 depicts the unemployment rate, our measure of inefficiency, as a function of \( \varepsilon \). It displays a U-shaped curve, with the random search allocation at the extreme left, and the directed search allocation at the extreme right. It shows two results. First, the directed search allocation is more efficient than the random search allocation. In the directed search allocation, the job-finding rate is 0.487 and the corresponding unemployment rate is 0.059. In the random search allocation, the corresponding figures are 0.465 and 0.062. That the job-seekers take into account the information vehicled by vacancy age is thus not only good for themselves but also for the society as a whole. Second, as expected from Proposition 4, the directed search allocation is not the most efficient one. The unemployment rate is minimized for \( \varepsilon = 0.25 \). Then, the job-finding rate is 0.498 and the unemployment rate is 0.058.
Figure 1: Unemployment rate as a function of $\varepsilon$. The random search allocation results when $\varepsilon=0$, whereas the directed search allocation results when $\varepsilon=1$.

Figure 2: Density of job-seekers by vacancy age (in months)
Figure 2 displays the density of job-seekers by vacancy age in three different allocations: random search ($\varepsilon = 0$), directed search ($\varepsilon = 1$) and the allocation that minimizes the unemployment rate displayed by Figure 1 ($\varepsilon = .25$). Figure 2 shows spectacular differences in terms of search behavior. In the random search allocation, the density slightly decreases over vacancy age: it halves in two months. In the directed search allocation, the density has a quasi spike for newly created vacancies. Then it strongly decreases. The almost efficient allocation reveals an intermediate behavior: the spike is much less pronounced for new vacancies. Figure 3 shows the resulting tightness by vacancy age in the three allocations. Tightness is constant under random search, whereas it strictly increases with vacancy age in the other cases. Job-seekers who care for phantom vacancies search more at low vacancy ages, and thus the ratio of advertised jobs to seekers increases over age.

To understand why directed search is inefficient, Figure 4 depicts the job-finding rate, whereas Figure 5 shows the nonphantom-job proportion. The job-finding rate is by definition constant under directed search, and strictly decreases over age in the other allocations. In these latter cases, reallocating individual search effort towards recent vacancies would increase the individual odds of finding a job. This individual gain would also lead to a social gain in the random search allocation. However, it would deteriorate welfare in the almost efficient allocation. Figure 5 tells why. That job-seekers search more for recently created jobs implies that the nonphantom
proportion decreases very fast with vacancy age. As phantoms haunt the search place for some time, such a strong decline in the nonphantom proportion lasts and affects all vacancy ages. The almost efficient allocation accounts for this effect. Thus the pace of phantom creation is smoother over vacancy age, and the nonphantom proportion declines less rapidly.

Directed search leads the job-seekers to over-apply for young vacancies. Reallocating some of the job-seekers from young vacancies to older ones can achieve efficiency gains. The reason for this outcome is that directed search implies a negative informational externality that goes through different vacancy ages. Prospecting jobs of a given age, say $a$, means finding jobs of such age. Phantoms result and they haunt the search market. However, because of the nature of directed search, phantoms only hurt agents who prospect jobs older than $a$. It follows that the magnitude of this externality decreases with vacancy age. When workers prospect newly created vacancies, they affect all the other agents through phantom creation. When they prospect much older vacancies, they affect almost none.

Quantitatively, efficiency gains achieved by the almost efficient allocation are rather modest. The magnitude of unemployment rate is only slightly affected by the way agents search for jobs. This is because any search strategy that produces more matches also leads to increase the stationary phantom stock. One way to see this is to compute the mean nonphantom proportion in the three allocations. We obtain $\pi_{rs} = 44.7\%$, $\pi_{ds} = 42.7\%$, $\pi_{eff} = 47.1\%$. This, however,
does not mean that phantom vacancies do not quantitatively impact the matching function. In this calibration, removing all phantoms from the market leads to massive employment gains. The corresponding allocation obtains with $\beta = 0$. The job-finding rate is 0.813 and the corresponding unemployment rate is 0.036.

We can compute the contribution of information obsolescence to overall frictions. In the absence of frictions, the unemployment rate would be $\min(1 - K, 0)$. In our calibration, $K = 0.972 < 1$. Thus the nonfrictional unemployment rate is equal to 0.028. Under directed search the unemployment rate is $u_{ds} = 0.059$. It follows that matching frictions account for half the unemployment rate. Without information obsolescence, the unemployment rate decreases to 0.038. Hence, phantom vacancies contribute to $0.021/0.031 \approx 68\%$ of overall frictions and $0.021/0.056 \approx 37\%$ of unemployment.

5 Extensions

We examine three extensions: vacancy renewal where job listings are randomly refreshed, Nash bargaining where the negotiated wage increases vacancy age, and the presence of lemon jobs that job-seekers cannot obtain.
5.1 Renewing offers

Websites offer the possibility to refresh offers. The vacancy age is then reset to zero. If the previous ad disappears, each vacancy is characterized by a single ad. If it does not, then a given vacancy has several ads at a time, each one corresponding to a specific age. The former effect is the pure effect of renewing offers. The latter effect mixes vacancy renewal with search intensity. We here focus on the former effect.

We assume that nonphantom vacancies are renewed at exogenous rate $\gamma$. This modifies the dynamics of vacancies as follows:

\[
\dot{v} = -M(a) - \gamma v(a), \tag{26}
\]

\[
v(0) = \lambda(1 - u) + \gamma v. \tag{27}
\]

At each age $a$, the number of vacancies decreases by the flow of matches formed with such jobs, and by the flow of vacancies that are renewed. The inflow of new vacancies is then equal to newly destroyed employment relationships $\lambda(1 - u)$ plus the total number of renewed vacancies $\gamma v$.

The rest of the model is unchanged. We thus proceed as in the previous section. We first derive the directed search allocation, and then construct the other possible allocations by manipulating the differential equation that governs tightness.

Under directed search, the allocation of job-seekers across the different vacancy ages leads to equalize the job-finding rate at the different ages. Thus $\mu(a) = \pi(a)m(1, \theta(a)) = m(1, \theta(0))$. The nonphantom proportion is now affected by vacancy renewal:

\[
\frac{\dot{\pi}}{\pi} = -(1 - \pi)(m(1, \theta)/\theta - \delta) - \beta \pi m(1, \theta)/\theta - \gamma (1 - \pi). \tag{28}
\]

The additional term $-\gamma(1 - \pi)$ reveals that vacancy renewal tends to speed up the decline of the nonphantom proportion with vacancy age. Renewing a vacancy means destroying a nonphantom vacancy to replace it by another one with a shorter age. This behavior atrophies the nonphantom proportion by age.

We deduce the dynamics of tightness:

\[
-\alpha(\theta_{ds}) \frac{\dot{\theta}_{ds}}{\theta_{ds}} = \left[ 1 - \frac{m(1, \theta_{ds}(0))}{m(1, \theta_{ds}(a))} \right] \left[ \delta \theta_{ds} - m(1, \theta_{ds}) \right] - \beta m(1, \theta_{ds}(0)) - \gamma \left[ 1 - \frac{m(1, \theta_{ds}(0))}{m(1, \theta_{ds}(a))} \right]. \tag{29}
\]

Vacancy renewal increases the slope of the tightness function. The consecutive decline in the nonphantom proportion must be compensated by a stronger increase in the job-to-unemployed ratio.
We proceed as in Section 3. We alter equation (29) as follows:

\[-\alpha(\theta_\varepsilon) \dot{\theta}_\varepsilon = \varepsilon \left\{ 1 - \frac{m(1, \theta_\varepsilon(0))}{m(1, \theta_\varepsilon(a))} \left[ \delta \theta_\varepsilon - m(1, \theta_\varepsilon) \right] - \beta m(1, \theta_\varepsilon(0)) \right\} - \gamma \left[ 1 - \frac{m(1, \theta_\varepsilon(0))}{m(1, \theta_\varepsilon(a))} \right]. \tag{30} \]

To find \( \theta_\varepsilon(0) \), we use the resource constraint \( 1 - u + v = K \) together with the motions for \( v(a) \) and \( p(a) \) and the definition of tightness \( u(a) = (v(a) + p(a))/\theta(a) \).

We calibrate the model as in Section 3. On top of the different parameters displayed by Table 1, we need to set \( \gamma \), the renewal rate. As a benchmark we set \( \gamma = 2.0 \): job listings are renewed every two weeks on average. The parameter \( m_0 \) adjusts so that the job-finding rate stays the same. This gives \( m_0 = 0.75 \). The distribution of job-seekers, the pattern of tightness and job-finding rate, and the nonphantom proportion by vacancy age are very similar to Figures 2 to 5.

Figure 6 shows that the results obtained in Section 3 are qualitatively unchanged. The relationship between the unemployment rate and parameter \( \varepsilon \) is U-shaped. The random search allocation is dominated by the directed search allocation. Moreover, the unemployment rate is minimized for \( \varepsilon = 0.7 < 1 \), i.e. the directed search allocation is not constrained efficient.

Vacancy renewal increases the magnitude of unemployment rate differences between the random search and the directed search allocations. The unemployment rate is 7.1% in the
former case, against 5.9% in the latter one. Thus the difference is 1.2 percentage points, against 0.2-0.3 percentage points when $\gamma = 0$. Random search is more costly with renewal: as explained below equation (28), the nonphantom proportion declines more rapidly with age.

The difference between the directed search and almost efficient allocations is small, about 0.1 percentage point. However, information obsolescence still plays an important role in the magnitude of unemployment. The nonfrictional unemployment rate is unchanged and equal to 0.028. Thus the contribution of matching frictions to unemployment stays the same, slightly above 50%. By setting $\beta = 0$ or $\delta = \infty$, the unemployment rate reaches 0.047. It follows that phantom vacancies explain $1.2/5.9 \approx 20\%$ of unemployment, and $1.2/3.1 \approx 39\%$ of overall matching frictions.

We finally examine the role played by the rate of vacancy renewal. Figure 7 shows the unemployment rate in the directed search allocation as a function of $\gamma$. It strictly decreases with $\gamma$. As $\gamma$ becomes very large, the unemployment rate tends to 0.047 where information obsolescence plays no role.

Figure 7: Unemployment rate in the directed search allocation as a function of parameter $\gamma$
5.2 Wage bargaining

The fixed-wage allocation is inefficient as job-seekers search for too young vacancies. To resolve this inefficiency, it is necessary to provide stronger incentive to search for older vacancies. In other words, the wage contract should be conditional on the vacancy age. We here focus on Nash bargaining over the match surplus.

Once a worker and a firm meet, they bargain a non-renegotiable wage \( w(a) \). Thus the worker is paid this wage until the dissolution of the match. In such a case, the worker joins the pool of unemployed and the job becomes a new vacancy, i.e. a vacancy of age \( 0 \).

We have

\[
\begin{align*}
    rU &= \max_{a \geq 0} \{ \mu(a)[W(a) - U] \}, \\
    rW(a) &= w(a) + \lambda[U - W(a)], \\
    rV(a) &= \eta(a)[J(a) - V(a)] + V'(a), \\
    rJ(a) &= 1 - w(a) + \lambda[V(0) - J(a)].
\end{align*}
\]

Let \( S(a) = W(a) - U(a) + J(a) - V(a) \) be the match surplus when the firm and the worker meet:

\[
rS(a) = 1 - \lambda S(a) + \lambda[V(0) - V(a)] - rV(a) - rU.
\]

It follows that \( S'(a) = -V'(a) \) and \( S(a) = S(0) + V(0) - V(a) \). The worker obtains the share \( \nu \in [0, 1] \) of the match surplus, whereas the firm obtains the remaining share \( 1 - \nu \). Combining the different equations we obtain

\[
rS(a) = 1 - \lambda S(0) - \eta(a)(1 - \nu)S(a) - rU + S'(a).
\]

In equilibrium all sub-markets are open and yield the same utility. This implies that \( \pi(a)m(1, \theta(a))S(a) = m(1, \theta_0)S(0) \) for all \( a \geq 0 \). Taking the derivative of the left-hand side with respect to \( a \) gives

\[
\frac{\dot{\theta}}{\theta} = -\frac{\dot{\pi}}{\pi} - \frac{\dot{S}}{S}.
\]

The change in tightness is equal to the opposite of the change in nonphantom proportion minus the change in match surplus. Thus if match surplus goes up with age, then the tightness-vacancy age profile is less steep than in the fixed-wage allocation. As the bargained wage increases with the vacancy age, job-seekers have stronger incentive to search for older vacancies.

Solving for the match surplus, we obtain

\[
S(a) = \frac{\int_a^\infty \exp \left[ - \int_a^b (r + (1 - \nu)\eta(s)) \, ds \right] db}{1 + (\lambda + \nu m(1, \theta_0)) \int_0^\infty \exp \left[ - \int_0^b (r + (1 - \nu)\eta(s)) \, ds \right] db}.
\]
After some computations,
\[
\frac{\dot{S}}{S} = \frac{(r + (1 - \nu)\eta(b)) \int_a^\infty (1 - \alpha) \frac{\delta}{(r + (1 - \nu)\eta(b))} \exp \left[- \int_a^b (r + (1 - \nu)\eta(s)) \, ds\right] \, db}{\int_a^\infty \exp \left[- \int_a^b (r + (1 - \nu)\eta(s)) \, ds\right] \, db},
\]
which shows that the match surplus increases with vacancy age when the tightness-vacancy age pattern is itself increasing.

An interesting implication is that Nash bargaining can help to internalize the vacancy-age dependent informational externality.

5.3 Lemons vs phantoms

In this paper, the only reason why workers pay attention to the vacancy age is because they fear phantom jobs. We now turn to another line of explanation based on the existence of lemon jobs. Like phantoms, lemons cannot be occupied and so they create congestion for the job-seekers. Ad duration conveys a signal on the probability that the job is a lemon. This implies that workers are more inclined to search for young vacancies. We now examine this argument in our framework.

Let \( l(a) \) be the number of lemons at age \( a \), whereas \( l = \int_0^\infty l(a) \, da \) is the total number of lemons. Lemon vacancies create the same type of noise as phantoms do. The number of matches at age \( a \) is

\[
M(a) = \frac{v(a)}{v(a) + p(a) + l(a)} m(u(a), v(a) + p(a) + l(a)).
\]

We now denote \( \pi(a) = v(a)/(v(a) + p(a) + l(a)) \) and \( \theta(a) = (v(a) + p(a) + l(a))/u(a) \)

At each time there is an inflow of new lemons \( l(0) = l_0 \). These lemons disappear at the same rate as phantom vacancies do. Thus \( \dot{\lambda} = -\delta l(a) \). To ensure that the number of lemons stays constant, we suppose \( l_0 = \delta l \). Alternatively, we could assume there is an exogenously given stationary distribution of lemons across vacancy ages, which density is non-increasing over age.

Under directed search, workers spread over the different vacancy ages so that the job-finding rate is the same at all ages. Thus \( \pi(a)m(1, \theta(a)) = \pi(0)m(1, \theta(0)) \). As in the phantom model, we have \( a\dot{\theta}/\theta = -\dot{\pi}/\pi \) with

\[
\frac{\dot{\pi}}{\pi} = \frac{m(1, \theta)}{\theta} - [(1 - \beta)\pi - 1] + \delta(1 - \pi).
\]

Using this equation with the condition \( \pi(a) = \pi(0)m(1, \theta(0))/m(1, \theta(a)) \) leads to

\[
a\dot{\theta} = -(1 - \beta)\pi(0)m(1, \theta_0) + m(1, \theta) - \delta \theta + \pi(0)\delta m(1, \theta_0)/m(1, \theta).
\]

To find \( \theta_0 \), we proceed as before. The unemployment rate is now \( u = \lambda/(\lambda + m(1, \theta_0)) \), whereas the number of vacancies is \( \pi(0)\lambda(1 - u) \int_0^\infty \exp[-\int_0^a m(1, \theta(s))/\theta(s)ds] \, da \). Finally, the resource
constraint is $1 - u + v = K$, which gives

$$\frac{m(1, \theta_0)}{\lambda + m(1, \theta_0)} \left\{ 1 + \pi(0) \lambda \int_0^\infty \exp \left[ - \int_0^a m(1, \theta(b))/\theta(b) db \right] da \right\} = K. \quad (41)$$

A directed search allocation is a function $\theta(a)$ solving the differential equation (40) such that $\theta(0) = \theta_0$ and the resource constraint (41) holds. The computation of equilibrium is slightly more demanding because there is an additional fixed-point problem. Indeed, the tightness pattern is conditional to $\pi(0)$, whereas this proportion must check $\pi(0) = \lambda(1 - u)/(\lambda(1 - u) + l_0)$.

As in the previous Sections, we do not limit the analysis to the directed search allocation. Instead, we consider a range of allocations indexed by $\varepsilon$:

$$\alpha \theta = \varepsilon \{ -(1 - \beta) \pi(0) m(1, \theta_0) + m(1, \theta) - \delta \theta + \pi(0) \delta \theta m(1, \theta_0)/m(1, \theta) \}.$$ 

The resource constraint is modified accordingly.

We consider a parameterization similar to Section 3, Table 1. We set phantoms to 0 by having $\beta = 0$. We choose $l_0$ so that the lemon proportion among new ads is 10% in the directed search allocation. As for their death rate, we consider two cases $\delta = .5$ and $\delta = 3$. We need an explicit assumption about the way lemons are accounted for in the JOLTS dataset. Like phantoms, we suppose that firms do not declare lemons in the vacancy set. It follows that $x = v/u = 0.5$. The parameter $m_0$ adjusts so that the predicted job-finding rate of the directed search allocation equals the US one.

We start with $\delta = 0.5$. Figure 8 shows the unemployment rate as a function of $\varepsilon$, whereas Figure 9 shows the equilibrium tightness-vacancy age schedule for three values of $\varepsilon$. As lemons disappear at a slower rate than vacancies, the lemon proportion increases with age in the directed search allocation. Thus tightness must increase with age to compensate for the rise in lemon proportion. The unemployment rate is minimized for an intermediate value of $\varepsilon \approx 0.7$. Like in the phantom case, market segmentation by age is associated to an informational externality that decreases with age. Searching for newly created vacancies increases the lemon proportion at all future ages, which is not accounted for by the job-seekers when they spread over the different market segments. This is why the tightness-vacancy age schedule is less steep when $\varepsilon = 0.7$ than when $\varepsilon = 1$.

The magnitude of the externality is small. The unemployment rate varies from 0.0568 at top to 0.0562 at the lowest. Matching does not generate the kind of intertemporal frictions that we study in the phantom case. Thus the extent of additional frictions is limited by the lemon proportion among newly created vacancies. We limit this proportion to 10%, and we do not see much larger numbers as being realistic.

Then we set $\delta = 3$. Figure 10 reveals that tightness decreases with vacancy age. This arises because lemons disappear at a faster rate than phantoms. Thus the lemon proportion is larger
Figure 8: Unemployment rate as a function of $\varepsilon$ in the lemon case. The random search allocation results when $\varepsilon = 0$, whereas the directed search allocation results when $\varepsilon = 1$. 
Figure 9: Tightness by vacancy age (in months) with lemons, case $\delta = 0.5$
Figure 10: Tightness by vacancy age (in months) with lemons, case $\delta = 3$
at younger vacancy ages and job-seekers search for older vacancies as result. This case cannot arise in the phantom case because the phantom proportion among newly created vacancies is equal to 0. Thus the phantom proportion cannot decrease over time. When the phantom death rate is very large, then the phantom proportion stays constant after a finite age.

6 Conclusion

To be added

References

[1] To be added
A Proofs

Proof of Proposition 1:
We have \( v = \int_0^\infty v(a)da \). Thus \( v = \lambda(1-u)\theta/m(1,\theta) \). The resource constraint \( 1-u+v = K \) implies that \( (1-u)(1+\lambda\theta/m(1,\theta)) = K \). When combined with (12), this equation gives (13).
To prove existence and uniqueness, let \( \psi : [0,\infty) \to (-\infty, \infty) \) be such that
\[
\psi(\theta) = m(1,\theta)(1 - K) + \lambda\theta - K\lambda\frac{\beta m(1,\theta)}{\theta} - K\lambda.
\] (42)
The steady-state tightness, if any, is such that \( \pi_{rs}(\theta) = 0 \). The properties of the function \( m \) imply that \( \lim_{\theta \to 0} \psi(\theta) < 0 \) and \( \lim_{\theta \to \infty} \psi(\theta) > 0 \). By continuity, there is \( \theta_{rs} > 0 \) such that \( \psi(\theta_{rs}) = 0 \). Moreover,
\[
\psi'(\theta_{rs}) = \frac{1}{\theta_{rs}} \left[ \frac{\alpha(\theta_{rs})m(1,\theta)(1 - K) + \lambda\theta + K\lambda\frac{\beta m(1,\theta)}{\theta}}{1 + \lambda\theta/m(1,\theta_{rs})} \right].
\] (43)
Using the fact that \( \psi(\theta_{rs}) = 0 \), it comes
\[
\psi'(\theta_{rs}) = \frac{1}{\theta_{rs}} \left[ \frac{\alpha(\theta_{rs})K\lambda + (1 - \alpha(\theta_{rs}))\lambda\theta_{rs} + K\lambda\frac{\beta m(1,\theta_{rs})}{\theta_{rs}}}{1 + \lambda\theta/m(1,\theta_{rs})} \right] > 0.
\] (44)
It follows that \( \theta_{rs} \) is unique.

Proof of Proposition 2:
A directed search allocation must be characterized by these two equations. Once the function \( \pi_{ds}(a) \) is known, we can compute \( \pi(a), v(a), p(a), \) and \( u(a) \). We now prove existence and uniqueness. At given \( \theta(0) = \theta_0 \), equation (22) determines a unique function \( \theta(a,\theta_0) \). This function strictly increases with \( a \). Let \( \psi : [0,\infty) \to (-\infty, \infty) \) be such that
\[
\psi(\theta_0) = \frac{m(1,\theta_0)}{\lambda + m(1,\theta_0)} \left[ 1 + \lambda \int_0^\infty \exp \left[ - \int_0^a \frac{m(1,\theta(b,\theta_0))}{\theta(b,\theta_0)} db \right] da \right] - K.
\] (45)
We have \( \lim_{\theta \to 0} \psi(\theta) = -K \), whereas \( \lim_{\theta \to \infty} \psi(\theta) = +\infty \). Thus there is \( \theta_0 > 0 \) such that \( \psi(\theta_0) = 0 \). The function \( \psi \) is strictly increasing. Therefore \( \theta_0 \) is unique.

Proof of Proposition 3:
We transform the constraints \((c2), (c5)\) and \((c6)\) into differential equations. We define the following functions:
\[
\phi(a) = \int_0^a [m(\pi(b)u(b), v(b)) + \lambda u(b)] db,
\] (46)
\[
\psi(a) = \int_0^a [v(b) - u(b)] db,
\] (47)
\[
\chi(a) = -\lambda \int_0^a u(b) db.
\] (48)
The optimization program (*) is equivalent to
\[ \max_{u(.)} \int_{0}^{\infty} u(a) da \]
subject to
\[ \dot{v} = -m(\pi(a)u(a), v(a)), \]  
\[ v(0) = v_0, \]  
\[ \phi'(a) = m(\pi(a)u(a), v(a)) - \lambda u(a), \]  
\[ \phi(0) = 0, \]  
\[ \phi(\infty) = \lambda, \]  
\[ \dot{\psi} = v(a) - u(a), \]  
\[ \psi(0) = 0, \]  
\[ \psi(\infty) = K - 1, \]  
\[ \dot{\chi} = -\lambda u(a), \]  
\[ \chi(0) = 0, \]  
\[ \chi(\infty) = v_0 - \lambda. \]

The Lagrangian is
\[ \mathcal{L}(.) = -u - \sigma_1 m(\pi u, v) + \sigma_2 [\beta m(\pi u, v) - \delta p] + \sigma_3 (m(\pi u, v) + \lambda u) + \sigma_4 (v - u) - \sigma_5 \lambda u \]

The optimality conditions are
\[ \frac{\partial \mathcal{L}}{\partial u} = -1 + \pi m_1 (-\sigma_1 + \beta \sigma_2 + \sigma_3) + \sigma_3 \lambda - \sigma_4 - \sigma_5 \lambda = 0, \]  
\[ \frac{\partial \mathcal{L}}{\partial v} = \left( \frac{u}{v} \pi (1 - \pi) m_1 + m_2 \right) (-\sigma_1 + \beta \sigma_2 + \sigma_3) + \sigma_4 = -\dot{\sigma}_1, \]  
\[ \frac{\partial \mathcal{L}}{\partial p} = -\frac{u}{v} \pi^2 m_1 (-\sigma_1 + \beta \sigma_2 + \sigma_3) - \sigma_2 \delta = -\dot{\sigma}_2, \]  
\[ \frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial \mathcal{L}}{\partial \chi} = 0 = \dot{\sigma}_3 = \dot{\sigma}_4 = \dot{\sigma}_5. \]

Equations (64) imply that \( \sigma_3, \sigma_4 \) and \( \sigma_5 \) are constant over age \( a \).

From equation (61), we have
\[ \pi m_1 (-\sigma_1 + \beta \sigma_2 + \sigma_3) = 1 + \sigma_4 + (\sigma_5 - \sigma_3) \lambda \equiv t. \]

We also know that
\[ m_1 = (1 - \alpha)m(1, \theta), \]  
\[ m_2 = \alpha m(1, \theta)/\theta, \]  
\[ \frac{u}{v} = \frac{u + p}{v + p} = \frac{1}{\theta} \pi. \]
We replace (65)–(68) in equations (62) and (63). We obtain

\[
\pi(1 - \alpha)m(1, \theta)(-\sigma_1 + \beta \sigma_2 + \sigma_3) = t, \\
\dot{\sigma}_1 = -\frac{1}{1 - \alpha} \frac{t}{\theta} \left(\alpha + \frac{p}{v}\right) - \sigma_4, \\
\dot{\sigma}_2 = \frac{t}{\theta} - \delta \sigma_2. 
\]  
(69)  
(70)  
(71)

Differentiating (69) with respect to \(a\), we get

\[
-\frac{t \dot{\alpha}}{1 - \alpha} + \alpha \frac{\dot{\theta}}{\theta} + t \frac{\pi}{\pi} (1 - \alpha) \pi m(1, \theta) [-\dot{\sigma}_1 + \beta \dot{\sigma}_2] = 0. 
\]  
(72)

We now assume that \(t \neq 0\). Using (69)–(71), and dividing by \(t\), we obtain

\[
-\frac{\dot{\alpha}}{1 - \alpha} + \alpha \frac{\dot{\theta}}{\theta} - \beta \frac{m(1, \theta)}{\theta} \pi + \left[\delta - \frac{m(1, \theta)}{\theta}\right] (1 - \pi) \\
+(1 - \alpha) \pi \frac{m(1, \theta)}{\theta} \left[\frac{1}{1 - \alpha} \left(\alpha + \frac{p}{v}\right) + \beta + \frac{\sigma_4 - \beta \delta \sigma_2}{\theta}\right] = 0. 
\]  
(73)  
(74)

Rearranging terms, noting that \(\dot{\alpha} = \alpha' \dot{\theta}\) and \(p/v = (1 - \pi)/\pi\), and denoting \(b = \sigma_4/t\) and \(y = \sigma_2/t\), we finally obtain (24). To get (25), we integrate the following differential equation:

\[
\dot{y} = \frac{1}{\theta} - \delta y, 
\]  
(75)

with \(y(0) = y_0\).

**Proof of Proposition 4:**

(i) Let \(\beta > 0\) and suppose that both allocations coincide. Then \(\dot{\theta}(a) = 0\) and \(\theta(a) = \theta_{rs}\) for all \(a \geq 0\). Equation (24) implies that

\[
b_0 = \alpha \frac{\beta - 1}{(1 - \alpha)\theta_{rs}}, \\
\frac{1 - \pi}{\pi} = \beta (1 - \alpha)my. 
\]  
(76)  
(77)

The term \((1 - \pi)/\pi = p/v\) can be computed using equations (9) and (10). The variable \(y(a)\) can be computed from (25). Replacing \((1 - \pi)/\pi\) and \(y(a)\) by the computed values, we obtain

\[
e^{-\delta a}(\sigma - \beta(1 - \alpha)my_0 + \beta(1 - \alpha)m/(\delta \theta)) - \sigma e^{-\eta a} - \beta(1 - \alpha)m/(\delta \theta) = 0. 
\]  
(78)

This equation must hold for all \(a \geq 0\). This is impossible unless \(\beta = 0\). In this case, \(\sigma = 0\) and equation (78) is always satisfied.

(ii) For convenience, we rewrite the differential equations that describe the efficient and the directed search allocations:

\[
[\alpha - g(\theta)]\dot{\theta}_e = -\pi m_e(\alpha + (1 - \alpha)b_0 \theta_e) - \delta (1 - \pi) \theta_e + \beta \pi m_e(\alpha + (1 - \alpha) \delta \theta_e), \\
\alpha \dot{\theta}_d = -\delta (1 - \pi) \theta_d + m_d - (1 - \beta)m_0. 
\]  
(79)  
(80)
with \( m_0 \equiv m(1, \theta_0) \), \( m_e \equiv m(1, \theta_e) \), \( m_{ds} \equiv m(1, \theta_{ds}) \) and \( g(\theta) \equiv \theta \alpha'(\theta)/(1 - \alpha(\theta)) \). Both allocations coincide iff \( \theta_{ds}(a) = \theta_e(a) \) for all \( a \geq 0 \) and \( \theta_{ds}(0) = \theta_e(0) = \theta_0 \). We now neglect the indices \( ds \) and \( e \). As \( \pi m = m_0 \) in the directed search allocation, we have

\[
[\alpha - g(\theta)]\dot{\theta} = -m_0(\alpha + (1 - \alpha)b_0\theta) - \delta(1 - m_0/m)\theta + \beta m_0(\alpha + (1 - \alpha)\delta y\theta), \quad (81)
\]

\[
\alpha \dot{\theta} = -\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0. \quad (82)
\]

Thus

\[(1 - \alpha)m_0\beta \delta y \theta = h(\theta)[-\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] + m_0(\alpha + (1 - \alpha)b_0\theta) + \delta(1 - m_0/m)\theta - \beta \alpha m_0, \quad (83)\]

with \( h(\theta) = 1 - g(\theta)/\alpha(\theta) \). Equation (83) provides a first relationship between \( y \) and \( \theta \). This relationship must hold for all \( a \geq 0 \). This implies that its differential with respect to \( a \) must be equal to 0. Hence,

\[
\dot{\theta}(h'(\theta)[-\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] + (h(\theta) - 1)[-\delta(1 - m_0/m) + \delta m_0 m'/m^2] + h(\theta)m'_\theta + m_0(1 - \alpha)b_0 \}
\]

\[
= (1 - \alpha)m_0\beta \delta (\theta \dot{y} + y \dot{\theta}). \quad (84)
\]

Using (75) and (82), we can replace \( \dot{y} \) and \( \dot{\theta} \) by functions of \( y \) and \( \theta \). This gives

\[
(1 - \alpha)m_0\beta \delta y \theta = (\dot{\theta}/\theta - \delta)^{-1} \times \]

\[
\{h'[\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] + (h - 1)(-\delta(1 - m_0/m) + \delta m_0 m'/m^2) + hm' + (1 - \alpha)m_0b_0\} - (1 - \alpha)m_0\beta \delta. \quad (86)
\]

This gives a second relationship between \( y \) and \( \theta \). We can eliminate \( (1 - \alpha)m_0\beta \delta y \theta \), which finally gives

\[
[-\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] \times \]

\[
\{h'[\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] + (h - 1)(-\delta(1 - m_0/m) + \delta m_0 m'/m^2) + hm' + (1 - \alpha)m_0b_0\} - (1 - \alpha)m_0\beta \delta \]

\[
= \{h[\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0] + m_0(\alpha + (1 - \alpha)b_0\theta) + \delta(1 - m_0/m)\theta - \beta \alpha m_0\} \times \]

\[
[(-\delta(1 - m_0/m)\theta + m - (1 - \beta)m_0)/\theta - \alpha \delta], \quad (87)
\]

which has the form \( f(\theta) = 0 \). The directed search and the efficient allocation coincide only if this equation is satisfied for all \( \theta(a), a \geq 0 \). When \( \beta > 0 \), \( \theta \) is continuously increasing in \( a \), and so the equation must hold for the whole subset \( [\theta_0, \infty) \). This is generally not the case, as the properties of \( f \) depend on those of \( h \), which is a primitive of the model. Equation (87) actually
defines a differential equation in $h$. This differential equation may have a solution, and this may be compatible with the requirement $0 \leq \alpha(\theta) \leq 1$ for all $\theta \in [0, \infty)$. We do not know whether this case is possible or not. However, $h$ is exogenous and so it has no reason to coincide with this particular function, if it exists. Consider for instance the Cobb-Douglas case with $\alpha(\theta) = \alpha$ for all $\theta \geq 0$. This gives $h(\theta) = 1$ and $h'(\theta) = 0$. After simplification, equation (87) is now

$$-(2 + \alpha(1 - \beta))\delta m_0 + \alpha \delta m_0 b_0 \theta + \delta m/(1 - \alpha) + 2(1 - \beta)m_0 m/\theta$$

$$-m^2/\theta + (1 - \beta)\delta m_0^2/m - (1 - \beta)^2 m_0^2/\theta$$

$$= 0$$

This equation has a finite number of solutions in $\theta \in [\theta_0, \infty)$ – none is also a possibility. However, it does not hold for all $\theta \in [\theta_0, \infty)$.

Finally, when $\beta = 0$, equation (87) is equivalent to $b_0 \theta_0 = -\alpha/(1 - \alpha)$. This condition is also the one that ensures $\theta_c(a) = \theta_{ds}(a) = \theta_{rs}$ for all $a \geq 0$.

**B Numerical simulations**

Let $\varepsilon$ vary from 0 to 1. For each $\varepsilon$, we find the function $\theta_{\varepsilon} : [0, \infty) \rightarrow [0, \infty)$ with typical element $\theta_{\varepsilon}(a)$ that solves

$$\alpha \dot{\theta}_{\varepsilon} = \varepsilon[-\delta \theta_{\varepsilon} + m_0 \theta_{\varepsilon}^{\alpha} + m_0 \theta_{\varepsilon}^{\alpha-1} - (1 - \beta)m_0 \theta_{\varepsilon}^\alpha].$$

(A)

At given $\theta_0$, the differential equation (A) defines a unique function $\theta_{\varepsilon}(a) \equiv \theta_{\varepsilon}(a, \theta_0)$. Then $\theta(0)$ solves the resource constraint $1 - u + v = K$.

For a given $\theta_{\varepsilon}(a)$, we numerically solve

$$d\tilde{v}/da = -\tilde{v}(a)m_0 \theta_{\varepsilon}(a)^{\alpha-1},$$

$$d\tilde{p}/da = \beta \tilde{v}(a)m_0 \theta_{\varepsilon}(a)^{\alpha-1} - \delta \tilde{p}(a),$$

$$d\tilde{u}/da = (\tilde{v}(a) + \tilde{p}(a))/\theta_{\varepsilon}(a)$$

with $\tilde{v}(0) = 1$, $\tilde{p}(0) = 0$ and $\tilde{u}(0) = 0$. Then we compute $u = \lambda(1 - u) \int_0^\infty \tilde{u}(a) da$. This gives

$$u = \frac{\lambda \tilde{u}(\infty)}{1 + \lambda \tilde{u}(\infty)}.$$  

(92)

The resource constraint is

$$1 + \lambda \int_0^\infty \tilde{v}(a) da = K,$$

which is equivalent to

$$1 - K + \lambda \int_0^\infty \tilde{v}(a) da - \lambda \tilde{u}(\infty) = 0.$$  

(94)
Let \( \psi : [0, \infty) \to [0, \infty) \) be such that
\[
\psi(x) = 1 - K + \lambda \int_0^\infty \tilde{v}(a) da - \lambda \tilde{u}(\infty).
\]
The algorithm of resolution is as follows. Fix a lower bound \( \theta_- \) and an upper bound \( \theta_+ \) to \( \theta_0 \). Fix also \( a_{\text{max}} \), the approximation for \( +\infty \), and \( \tau > 0 \) a (small) tolerance parameter.

Step 1. Pick \( x = (\theta_- + \theta_+)/2 \), and solve the differential equation (A) from \( a = 0 \) to \( a = a_{\text{max}} \).

Step 2. Compute \( \psi(x) \). If \( |\psi(x)| < \tau \), go to Step 4. If not, go to Step 3.

Step 3. If \( \psi(x) < 0 \), set \( \theta_+ = x \). Else, set \( \theta_- = x \). Go to Step 1.

Step 4. Set \( \theta_0 = x \) and compute the different functions used in the text.

Proposition 3 describes the efficient tightness as follows:
\[
\begin{align*}
\alpha \dot{\theta} &= -\pi m_0 \theta^{\alpha-1} [\alpha + b_0 (1 - \alpha) \theta] \\
&\quad + \delta (1 - \pi) - \beta \pi m_0 \theta^{\alpha-1} [\alpha + \delta y (1 - \alpha) \theta], \quad (\text{A'}) \\
\dot{y} &= 1/\theta - \delta y, \quad (\text{A''})
\end{align*}
\]
where parameters \( b_0 \) and \( y_0 \) are still to be found. The initial parameter \( \theta_0 \) must be such that the resource constraint \( 1 - u + v = K \) holds.

We proceed as follows. We determine a closed subset \( S \subset [0, \infty) \times [0, \infty) \) of possible values of \( b_0 \) and \( y_0 \). We then set a grid within this subset, where each point of the grid is an element of \( S \), and we go from one element to the other by means of two indices, one for the line and the other for the column. For each pair \( (b_0, y_0) \), we find \( \theta(a) \) with the previous algorithm where we replace equation (A) by equations (A')-(A''). Then, we pick the lowest unemployment rate of the grid, and the corresponding allocation is the efficient one.