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Heterogeneous Agents and General Equilibrium in Financial Markets

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Abstract

This paper investigates the effects of heterogeneous preferences on risk premia, on asset price dynamics, and on distributions of wealth and consumption. I consider a continuous time economy populated by an arbitrarily large number of agents whose CRRA preferences differ in their risk aversion parameter. I derive closed form expressions for the interest rate, the market price of risk, and drift and diffusion of consumption weights. A closed form expression for asset prices being unattainable, I derive an estimable expectation formula for the price of a risky stock. I then simulate the economy using a parallelized/GPU implementation of path monte-carlo methods to estimate asset prices and compare the qualitative and quantitative features to real world observations. The model does well in replicating a falling dividend to price ratio over time, low risk free rate, a high risk premium, and rising inequality. I also study the effects of increasing the number of agents and the level of disagreement and find that more agents and lower disagreement, corresponding to a more equitable initial condition, produces lower individual leverage, lower asset prices, and higher expected excess returns on risky assets.

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Introduction

Macroeconomic theory is evolving from a representative agent focus to a heterogeneous, agent-based one. When we study many agents we are implicitly stating that we believe there exists emergence in economics. By emergence, I refer to the definition of Miller, Page, and LeBaron (2008): "... that individual, localized behavior aggregates into global behavior that is, in some sense, disconnected from its origins." When the characteristics of a system as a whole cannot be explained as the sum of its parts, we say it exhibits emergent properties In particular, we believe that differences between agents, whether differences caused by situation, information, education, or inherent characteristics, create economic outcomes that cannot be predicted through the simple study of a single agent. One example of emergence is that of exchange. With a single agent, there can be no exchange. When it comes to financial markets, this shortcoming of the representative agent is never more apparent. Each day, trillions of dollars worth of financial assets change hands.

When pricing a financial security, one must answer the question, what is a security in the first place? Being simply a piece of paper, a financial security gives its bearer the right to a stream (not very continuous) of future dividends and capital gains for the (possibly) infinite future. The price of this abstract object is so difficult to determine that if you ask two analysts for an exact price they will generally disagree. In fact, even two equally informed analysts, in the exact same post, might disagree. It is this very disagreement that underlies this paper. The fact that two people in the same situation might price a risky asset differently could only be explained by differences in their preference towards risk. When we allow there to exist inherent differences in the way that agents price risk, we often end up with constantly shifting distributions of wealth and consumption, as well as changing financial variables. Although these characteristics make models quite difficult to solve, these are the very characteristics that bring them closer to the real world.

Since Dumas (1989), there has been both theoretical and empirical work on heterogeneous risk preferences. The field lay relatively dormant until Coen-Pirani (2004) studied the introduction of Epstein-Zin preferences into an economy of two agents and found the counterintuitive result that, under certain parameterizations, the most risk averse agent could dominate. Since then, the empirical work has focused on trying to estimate the distribution of risk aversion that might prevail in the economy, as in Kimball, Sahm, and Shapiro (2008). Theoretical work has tried to identify how and why different agents with different preferences participate in different ways. Guvenen (2006) uses participation constraints to study the effects of allowing only low risk aversion agents to participate in financial markets and are able to replicate well the real world wealth distribution. In a more complete markets vain, Bhamra and Uppal (2014), Chabakauri (2013), Gârleanu and Panageas (2008), and Cozzi (2011) have all worked on studying the interactions of two heterogeneous agents.

The majority of the theoretical work on heterogeneous risk preferences focuses on agent survival, asymptotic properties, and attempts to derive closed form solutions for asset prices when there are two agents, assuming that the results generalize to many agents. These techniques often miss the dynamics of the market and the complexity one would find using an economy populated by many heterogeneous agents. I try to fill this gap in the literature by deriving expressions for the dynamics of agents' consumption, interest rates, the market price of risk, and an estimable expression of asset prices. Using these I simulate an economy, studying its properties. My work most closely matches that of Cvitanić, Jouini, Malamud, and Napp (2011), who study an economy populated by agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. However, they focus on issues of long run survival and price, while I am more interested in simulating the short run dynamics of the model. Also, focusing on a single dimension of heterogeneity makes the conclusions more clear and the formulation less intimidating.

Apart from focusing on preferences, heterogeneous agent models take on several forms. One approach has been to assume agents' differ in their discount rate. Hendricks (2007) finds that indeed heterogeneity in discount rates can explain some of the wealth inequality we observe in the market. Possibly the most developed heterogeneous agent models are those involving idiosyncratic risk, as in Aiyagari (1994). This type of heterogeneity was studied by Krusell and Smith (1998) as a cause of the real world distribution of wealth. A third type of heterogeneity is that of heterogeneous beliefs as in Blume and Easley (1992) or Bullard and Duffy (1999). These types of models focus on the dynamics of the economy and survival rates. One of the unifying factors in all of these models is some form of market imperfection, be it imperfect insurance markets, imperfect financial markets, or imperfect information, that drives exchange. This seems unsatisfactory, since it is easy to think of an example where two people with the same information and the same budgets might disagree on the price of some risky asset. A promising way to allow for this type of disagreement is through heterogeneous preferences.

Another example of agent heterogeneity is that of heterogeneous financial constraints or frictions. Articles such as Mankiw and Zeldes (1991) look at how limited participation in financial markets can help explain the equity premium, but the participation assumption is often exogenous. Heterogeneous preferences allows this participation decision to be made endogenous. In fact, the formulas for the risk free rate and the market price of risk derived in this paper resemble greatly those in Basak and Cuoco (1998), where one of two agents is restricted from participating in the stock market. If we think of individual agents as each having a supply and demand function for risky assets and risk free bonds, it is possible to think of a model of heterogeneous risk preferences as one of market break down. Each agent populates a single, theoretical market, but only one market can clear.

The market which clears is the one corresponding to the agent who is indifferent between buying or selling their shares or bonds. However, contrary to the limited participation literature, the clearing markets for stocks and bonds do not have to correspond to the same agent and, furthermore can vary over time.

The addition of more agents attempts to explain certain shortcomings of the standard, representative agent framework. The explicit study of general equilibrium in financial markets, ushered in by Arrow and Debreu (1954), has progressed along a line of inquiry centered around a representative agent since the publication of the seminal paper by Lucas (1978). However, the representative agent framework fails in many respects. First, there is the infamous equity premium puzzle of Mehra and Prescott (1985), in which a representative agent needs to be incredibly risk averse to generate a real-world equity risk premium. At the same time, a high risk aversion implies a high risk free rate, creating the risk free rate puzzle of Weil (1989). Debate abounds about the "true" value of a representative agent's risk aversion parameter, but most work centers on a level around $2-3$ (e.g. Barro and Jin (2011)). Second, there is the observed phenomenon of positive trading volume. In a representative agent economy with perfect financial markets, trading volume is zero. Third, there is the existence of skewed distributions of income and wealth, documented in works on inequality like Saez and Zucman (2014) and Piketty and Saez (2001). Fourth, there is the leverage cycle documented by Geanakoplos (2010), where collateral constraints augment the fluctuations in asset prices given a negative shock. However, as with many other economic models, the representative agent has been a springboard for researchers to attempt to explain these phenomena through modifications and additions to the assumptions.

There have been several approaches taken in the literature to improve the explanatory power of the basic representative agent framework. One approach has been to allow for variable risk aversion parameters, using habit formation as in Campbell and Cochrane (1999) or Abel (1990), or using hyperbolic absolute risk aversion as in Ogaki and Zhang (2001). However, conflicting evidence has been found by Chiappori and Paiella (2011) who find that individuals' relative risk aversion is constant over time. Another approach is to introduce imperfections into the financial market. Research such as that done by Sharpe (1994) find significant impact of financial frictions on the real economy, while theoretical papers such as Bernanke, Gertler, and Gilchrist (1999) have studied the mechanism driving the effects of frictions.

In this paper I will develop a continuous time model of a Lucas economy, populated by an arbitrarily large number of agents who differ in their rate of relative risk aversion. I will then solve the model and simulate its dynamics. The structure of the paper is as follows: section 1 sets up and solves the model, section 3 give some simulation results, and section 4 concludes. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

In this section, I will describe the general setting of the model. The key components are the definition of agent heterogeneity, the economic uncertainty, agent optimization, and equilibrium. The solution method will be discussed in the following section.

1.1. Agents

I consider a continuous time economy populated by a finite number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. Each agent has constant relative risk aversion (CRRA) preferences such that their risk aversion parameter, γ_i, is drawn from a distribution $f(\gamma)$ whose support is open and bounded such that $\gamma_i \in (\underline{\gamma}, \overline{\gamma})$:

$$U_i(c(t)) = \frac{c(t)^{1-\gamma_i} - 1}{1 - \gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

$$\gamma_i \sim f(\gamma)$$

This form of agent heterogeneity creates difficulties in solving the model analytically. The system of equations generated tends to be highly non-linear. However, the use of the continuous time model allows us to use Itô's lemma as a form of second order linearization, maintaining preferences towards risk while reducing the complexity of the problem. Also, the definition of the distribution of risk aversion allows for a high degree of freedom in simulations. This model encompasses a representative agent, a two agent frame-work, and up to a countable, finite number of agents (although solution is limited by computational power).

1.2. Uncertainty

Agents have available to them one risky asset, representing claims to a Lucas tree whose dividend process follows a geometric Brownian motion, and risk free borrowing and lending at an interest rate $r(t)$ in zero net supply. All uncertainty in the model is driven by a standard Wiener process, $W(t)$, defined on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}_t)$. Thus the evolution of dividends in the economy is given by

$$\frac{dD(t)}{D(t)} = \mu_D dt + \sigma_D dW(t) \tag{1}$$

Agents continuously trade in claims to the dividend process. We will hypothesize that the share price, $S(t)$, also follows a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dW(t) \tag{2}$$

One of the main goal of this paper is to determine the values of $\mu_S(t)$ and $\sigma_s(t)$. If we think about a representative agent economy, we would expect the price-dividend ratio to be constant over time, and thus the drift and diffusion coefficients for the stock price are simply those of the dividend process. We should look in our analysis to see how this model differs and is similar in this respect to the representative agent framework.

1.3. Individual Optimization

All agents are initially endowed with a share in the tree, which I will assume to be uniformly distributed across agents, but this assumption can be easily relaxed and has little effect on the dynamics of the model. Agents choose simultaneously their consumption and portfolio choices. As a standard simplification, I will define the model in terms of wealth. At any time t an agents budget constraint can be written as

$$dX^i(t) = \left[r(t)X^i(t) + a^i(t)S(t)\left(\mu_S(t) + \frac{D(t)}{S(t)} - r(t)\right) - c^i(t)\right] dt + a^i(t)\sigma_s(t)S(t)dW(t) \tag{3}$$

where the set of variables $\{c^i(t), a^i(t), X^i(t), D(t), S(t), r(t), W(t)\}$ represent an agent's consumption, asset holdings, and wealth (defined as $X^i(t) = a^i(t)S(t) + b^i(t)$, $b^i(t)$ being bond holdings), as well as the dividend, market clearing asset price, market clearing risk free interest rate, and Wiener process governing the Brownian motion. The budget constraint is standard and can be interpreted as the instantaneous change in wealth being equal to the return on savings plus the net returns on holding the risky asset plus a mean zero random term governed by the risk process. Using these facts we can write an individual agent's constrained maximization subject to instantaneous wealth and dividend processes as:

$$\begin{array}{ll} \max\limits_{\{c^i(u), a^i(s), b^i(u)\}_{u=t}^{\infty}} & \mathbb{E}\displaystyle\int_t^{\infty} e^{-\rho(u-t)} \frac{c^i(u)^{1-\gamma_i} - 1}{1-\gamma_i} du \\ \text{s.t.} & eq.\ (1),\ eq.\ (2),\ \&\ eq.\ (3) \end{array}$$

1.4. Equilibrium

Definition 1. *An equilibrium in this economy is defined by a set of processes* $\{r(t), S(t), \{c^i(t), X^i(t), a^i(t)\}\}_{i=1}^N\}$ $\forall\ t$, *given preferences and initial endowments, such that* $\{c^i(t), a^i(t), X^i(t)\}$ *solve the agents' individual optimization problems and the follow-*

ing set of market clearing conditions is satisfied:

$$\begin{aligned}\sum_i c^i(t) &= D(t)\\ \sum_i a^i(t) &= 1 \\ \sum_i X^i(t) &= S(t)\end{aligned} \tag{4}$$

The definition of equilibrium can be interpreted as agents choosing their consumption and portfolio choices to maximize their individual utilities. These choices in turn must satisfy the market clearing conditions for consumption, asset holdings, and wealth. In order to do so, interest rates and asset prices adjust.

2. Equilibrium Characterization

This section will derive a solution to each agent's maximization problem and give results on the characteristics of equilibrium. In deriving the equilibrium, it is interesting to note that the choices of agents can be seen as determining first and foremost their consumption, which in turn determines their portfolio choice. These choices give rise to a market clearing price of risk, a risk free interest rate, and a stock price. All of these outcomes can be compared (see appendix A) to an economy populated by a single, representative agent.

2.1. Financial Markets

To solve an agent's optimization we can discount future consumption to today's value. We do so by multiplying by a stochastic discount factor process. Given that there is only one market and one risky asset, there is only one stochastic discount factor for the market. Following Karatzas and Shreve (1998) we can define the stochastic discount factor as

$$H_0(t) = exp\left(-\int_0^t r(u)du\right) exp\left(-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta(u)^2 du\right) \tag{5}$$

where

$$\theta(t) = \frac{\mu_s(t) + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)} \tag{6}$$

represents the market price of risk. This implies that the dynamics of the the stochastic discount factor also follow a geometric Brownian motion:

$$\frac{dH_0(t)}{H_0(t)} = -r(t)dt - \theta(t)dW(t) \tag{7}$$

It is important to keep in mind that agents are not discounting at their own rate, but at a market rate. This is because each agent knows that their only choice is to buy or sell assets at the market rate, assuming that there is no arbitrage. If it were possible for there to be many markets, each one clearing at an individual agent's price, one could buy risky assets in a market with a risk averse agent and sell them in a market with a risk neutral agent at a higher price, making a positive profit.

2.2. The Static Problem

Using the stochastic discount factor, defined in (5), we can rewrite each agent's dynamic problem as a static one

$$\begin{aligned} \max_{\{c^i(u)\}_{u=t}^{\infty}} \quad & \mathbb{E}\int_t^{\infty} e^{-\rho(u-t)} \frac{c^i(u)^{1-\gamma_i}-1}{1-\gamma_i} du \\ \text{s.t.} \quad & \mathbb{E}\int_t^{\infty} H_0(u)c^i(u)du \leq X_0^i \end{aligned}$$

If we denote Λ_i as the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c^i(u) = \left(e^{\rho u}\Lambda_i H_0(u)\right)^{\frac{-1}{\gamma_i}} \tag{8}$$

which holds for every agent in every period. Note here that, because the problem is static, there is only one Lagrange multiplier, Λ_i.

2.3. Consumption Weights

Given each agent's first order conditions, we can derive an expression for consumption as a fraction of total dividends.

Proposition 1. *One can define the consumption of individual, i, at any time, t, as a share $\omega^i(t)$ of the total dividend, $D(t)$, such that*

$$c^i(t) = \omega^i(t)D(t) \tag{9}$$

$$\textit{where } \omega^i(t) = \frac{\left(\Lambda_i e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^N \left(\Lambda_j e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_j}}} \tag{10}$$

This expression is very similar to one derived in Bhamra, Coeurdacier, and Guibaud (2014) where two agents face imperfectly integrated financial markets. There, the stochastic discount factor is equally weighted by the two log agents, but it differs thanks to the introduction of a tax. Here, the value of the stochastic discount factor is equal across agents, but differs in its weight for each agent as they differ in risk aversion. This leads one to think that perhaps it would be better to think of this as an incomplete market. If markets were fully complete, there would be a risky asset for each agent, but here agents are forced to bargain over a single asset. Also, since in this economy agents can only consume their share of the dividend process, which in not tradeable, their consumption weight is equal to their asset holdings, i.e. $\omega^i(t) = a^i(t)$.

To derive an expressions for a risk free rate and the market price of risk, it is necessary to derive a quick lemma about the drift and diffusion of agents' consumption processes:

Lemma 1. *If we model an agent's consumption as a geometric Brownian motion with time varying drift and diffusion coefficients $\mu_{c^i}(t)$ and $\sigma_{c^i}(t)$, then for a given $r(t)$ and $\theta(t)$, and for all $i \in \{1, ..., N\}$*

$$\begin{aligned} \mu_{c^i}(t) &= \frac{r(t) - \rho}{\gamma_i} + \frac{1 + \gamma_i}{\gamma_i^2} \frac{\theta(t)^2}{2} \\ \sigma_{c^i}(t) &= \frac{\theta(t)}{\gamma_i} \end{aligned}$$

Notice first that the diffusion process for an agent is equal to the market price of risk scaled down by risk aversion. This makes sense as we would expect risk averse agents to desire a lower volatility in consumption. Also notice that the drift coefficient is made up by two terms. The first term incorporates the agent's time preference: if the interest rate is higher than the agents' rate of time preference, this will increase the growth rate in consumption, as they see a gain in saving today to consume tomorrow. However, this term is again scaled down by risk aversion, as CRRA preferences have the characteristic that risk aversion is the inverse of the elasticity of inter-temporal substitution. Thus, the higher an agent's risk aversion, the lower their elasticity of inter-temporal substitution, and the less they are willing to adjust their consumption choices in the face of changes in the interest rate. The second term incorporates preferences towards risk. If an agent's risk aversion is greater than 1, this second term will be less than 1. This implies that changes in the market price of risk cause a proportionally smaller change in the growth rate in individual consumption. Conversely, if an agents risk aversion is less than 1, this term will be greater than 1, implying that changes in the market price of risk have an amplified effect on their consumption growth. Also, this term is decreasing in γ_i, implying that the higher the risk aversion, the lower the effects of changes in the market price of risk on the growth rate in an agent's consumption.

Given Lemma 1, we can derive expressions for the market price of risk and the risk free rate:

Proposition 2. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ *and* $\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$ *such that*

$$r(t) = \rho + \frac{\mu_D}{\xi(t)} - \frac{1}{2}\frac{\xi(t) + \phi(t)}{\xi(t)^3}\sigma_D^2 \tag{11}$$

$$\theta(t) = \frac{\sigma_D}{\xi(t)} \tag{12}$$

by Lemma 1 and eq. (4)*.*

Proposition 2 is in terms of only certain moments of the joint distribution of consumption and risk aversion. We can compare these formulas to those for any individual agent (see appendix A). In (12), we can see that the market price of risk in the heterogeneous economy is equal to the market price of risk that would prevail in a representative agent economy populated by an agent whose elasticity of inter-temporal substitution is equal to the consumption weighted average in our economy. This is driven by the fact that individual agents choose their risky asset holdings such that their private valuation of the entire economy equals the valuation of all other agents. However, they hold only a portion of the total market portfolio, and so they have a limited weight in the outcome.

Looking at (11), the first two terms are very reminiscent of the interest rate in a representative agent economy populated by the same agent that would determine the market price of risk. However, the last term of (11) is slightly different. If it were the case that $\phi(t) = \xi(t)^2$, then the interest rate and market price of risk in this model could be exactly matched by those in an economy populated by a representative agent with time varying risk aversion, as in the model of habit formation by Campbell and Cochrane (1999). However, in numerical simulations this is not the case (see section 3) and we tend to have $\phi(t) > \xi(t)^2$. This is driven by the non linearity in the elasticity of inter-temporal substitution as a function of risk aversion (namely that $\mathbb{E}\frac{1}{x} \neq \frac{1}{\mathbb{E}x}$). This causes the second term to be larger, and the risk free rate thus to be lower, than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge" between the price of risk and the price for risk free borrowing. As we'll see in section 3, this fact takes a step towards explaining the equity risk premium puzzle of Mehra and Prescott (1985) and the risk free rate puzzle of Weil (1989), but does not solve them entirely.

2.4. Consumption Weight Dynamics

We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Proposition 1.

Proposition 3. *Assuming consumption weights also follow a geometric Brownian motion such that*

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t)$$

an application of Itô's lemma to (10) *gives expressions for* $\mu_{\omega^i}(t)$ *and* $\sigma_{\omega^i}(t)$*:*

$$\mu_{\omega^i}(s) = (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \theta(t)^2\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right] \tag{13}$$

$$\sigma_{\omega^i}(t) = \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) \tag{14}$$

The expressions in Proposition 3 describe how an agent's position relative to the consumption weighted average of elasticity of inter-temporal substitution (and its square) in the economy determines the growth rate in their relative share in consumption and the covariance of that share with the risk process. For this analysis, define γ^* such that $\xi(t) = \frac{1}{\gamma^*}$.

Consider first (14), which describes how an agent's consumption weight co-varies with the risk process. If an agent i is more risk averse than the weighted average, that is $\gamma^i > \gamma^* \Leftrightarrow \frac{1}{\gamma^i} < \frac{1}{\gamma^*} = \xi(t)$, then (14) < 0 and agent i's weight is negatively correlated to the market. This implies that if an agent is more risk averse than the average then their asset holding will increase when there are negative shocks and decrease when there are positive shocks. This is a prudence motive and these agents can be thought of as playing a "buy low, sell high" strategy. Conversely, if an agent is less risk averse than the average their asset holdings covary positively with the market. These agents are essentially buying high and selling low, a strategy that in the long run will leave them under water in terms of wealth (see section 3). An agent with a lower risk aversion has a higher elasticity of inter-temporal substitution and, thus, can be thought of as less patient. Given a shock to the dividend process, the expected growth rate remains constant, but level shifts permanently because of the martingale property of the Brownian motion. Since less patient agents see the current output of the dividend as more important than its long-run behavior present shocks have a greater effect on their personal price. Thus, a negative shock causes them to reduce their price and in turn their asset holdings.

Consider next (13), which describes how an agent's consumption weight grows over time. The first term is the product of two separate terms: one involving the interest rate and rate of time preference, the other the agent's position in the distribution. If the interest rate is above the rate of time preference, the first term is positive. This is intuitive, as if the agent is being compensated through $r(t)$ by more than their time discount rate, ρ, then they should desire to transfer wealth to tomorrow and their growth rate will be larger. This is reminiscent of the equation of motion for capital in a standard Ramsey model, but multiplied by a function of individual risk aversion. If the agent is more risk averse than the weighted average, as in the analysis of eq. (14), then the product will be negative and this first term will contribute negatively to their growth rate $\mu_{\omega^i}(t)$. The combined effect of these two terms is to say that if an agent is less risk averse than the average and the interest rate is greater than their rate of time preference, they will want to grow their consumption faster than the rate of growth in the economy, while if they are more risk averse than the average then they will grow their consumption more slowly than the rate of growth in the economy. This, again, is driven by the elasticity of inter-temporal substitution and a prudence motive. More risk averse agents have lower elasticity of inter-temporal substitution and they are thus less willing to accept fluctuations in their consumption. Because of this, if the interest rate is greater than their rate of time preference they will desire more to divest from the risky assets and accumulate risk free bonds.

The second term is quite a bit more complex. The term in brackets is a sort of quadratic in deviations from the weighted average of risk aversion. Whether this term is positive or negative depends in a complicated way on $\xi(t)$ and $\phi(t)$ [1]. It is sufficient to note that, when the distribution is not too skewed, there exists a level of risk aversion such that if an agent is above this the second term in (13) is negative and that this level of risk aversion is not equal to $\gamma*$. Thus at any point in time there will be three groups: one whose drift is strictly positive, one where it is ambiguous, and one where it is strictly negative. The first group is willing to invest no matter the risks, the second is prudent but still willing to grow their portfolio, and the third would like to divest from the risky stock. Finally, as $\xi(t)$ and $\phi(t)$ converge toward each other in the long run, only the least risk averse agent will have a positive drift (again see section 3).

2.5. Asset Prices

Now, given that we have found expressions to describe the evolution of consumption choices over time, we can give a formula describing asset prices. Bear in mind that it is not trivial to solve for each individual agent's asset price, as it depends in a non-linear way on their consumption weight. However, given that we know how consumption will evolve we can use market clearing to derive the expression in Proposition 4.

Proposition 4. *Under a transversality condition on wealth, that is if we assume*

[1] It can be shown that the roots of this quadratic are $\frac{1}{\gamma_i} = \frac{2\xi(t)-1\pm\sqrt{1+4\phi(t)-8\xi(t)^2}}{2}$. Taking only the positive root, we have $\frac{1}{\gamma_i} = \xi(t) - \frac{1}{2} + \sqrt{\frac{1}{4} + \phi(t) - 2\xi(t)^2}$. Under certain distributions of $\omega^i(t)$ there is no real root and the term is strictly positive.

$\mathbb{E}_t\left[\lim_{s\to\infty} H_0(s)X^i(s)\right] = 0$, *then it can be shown that asset prices satisfy the following:*

$$S(t) = \mathbb{E}_t \int_t^{\infty} \frac{H_0(u)}{H_0(t)} D(u)du \tag{15}$$

Proposition 4 defines asset prices today in terms of expectations of the future outcome of the dividend process, discounted at the market rate. Consider (15), which we can rewrite by substituting for $H_0(t)$ using (8) as

$$S(t) = \mathbb{E}_t \int_t^{\infty} e^{-\rho(u-t)} \left(\frac{c^i(u)}{c^i(t)}\right)^{-\gamma_i} D(u)du$$

for every i, which is exactly equal to the asset pricing formula derived from the Euler equation in a representative agent economy. Notice, however, that the dynamics of the consumption process causes the two to differ. It may be possible to construct an agent whose consumption share remains fixed over a short period of time and thus whose price in a representative agent economy would equal the price in the heterogeneous economy, but it is not necessarily true that that agent exists in the model being solved here. For instance, in the case of two agents, the price will always be somewhere between the price that would prevail in the two individuals' autarkic economies.

The last proposition involves leverage. We are interested in the evolution of wealth, which is made up of risky assets and bond holdings. So, we should study how an individual's bond holdings and economy-wide borrowing evolve over time. First, we need a lemma about individual drift and diffusion coefficients.

Lemma 2. *Individual savings, $b^i(t)$, have the following dynamics:*

$$db^i(t) = \mu_{b^i}(t)dt + \sigma_{b^i}(t)dW(t)$$

where

$$\mu_{b^i}(t) = r(t)b^i(t) - \omega^i(t)S(t)(\mu_{\omega^i}(t) + \sigma_S(t)\sigma_{\omega^i}(t)) \tag{16}$$

$$\sigma_{b^i}(t) = -\sigma_{\omega^i}(t)\omega^i(t)S(t) \tag{17}$$

First, consider (17) and recall our analysis about $\sigma_{\omega^i}(t)$. If an agent is more risk averse than the weighted average, then their diffusion for consumption weight is negative, and thus their bond holdings co-move positively with the market. These risk averse agents are thus divesting from the risky asset and investing in risk free bonds as prices rise. When prices fall, they do the opposite, selling their bonds to purchase risky assets. The opposite is true of an agent who is less risk averse than the weighted average. Following a positive shock, agents with low risk aversion desire to increase their holdings of the risky asset to profit from the higher dividends being paid. To do so, they will leverage their position and will thus borrow more. In this way, agents with low risk aversion are financing their portfolio growth by borrowing on the bond market. We will see in section 3 that as the low risk aversion agents dominate, their price dominates the market as well, causing prices to rise. So, their borrowing is indirectly causing rising asset prices.

Next, consider (16). An agents savings increases with their returns on savings, $r(t)b^i(t)$, and falls with the change in their risky asset holdings. The change in their risky asset holdings is equal to the value of their holdings, $\omega(t)S(t)$, times the drift in asset holdings, $\mu_{\omega^i}(t)$, plus the covariance in their asset holdings and the stock price. Here it is important to note that the drift and diffusion of bond holdings are scaled by the price of the asset, so if there is a precipitous rise in asset prices, this has a scaling effect on borrowing. An agent whose bond holdings are low and has a high exposure to asset prices could move to a negative drift in their bond holdings that becomes reinforcing, as the first term now drags down the drift even further. In this way, it is possible for agents in this economy to fall into debt spirals where they borrow to pay off old debt and their wealth falls precipitously.

Now, we can use Lemma 2 to study the evolution of total borrowing and total financial leverage in the economy.

Proposition 5. *If we define total borrowing in the economy as* $\Gamma(t) = \frac{1}{2}\sum_{i=1}^{N} |b^i(t)|$ *and total financial leverage as* $L(t) = \frac{\Gamma(t)}{S(t)}$*, then it can be shown that these processes have the following dynamics:*

$$\begin{aligned} d\Gamma(t) &= \mu_\Gamma(t)dt + \sigma_\Gamma(t)dW(t) \\ \frac{dL(t)}{L(t)} &= \mu_L(t)dt + \sigma_L(t)dW(t) \end{aligned}$$

where

$$\begin{aligned} \mu_\Gamma(t) &= \frac{1}{2}\sum_{i=1}^{N}\left[sgn(b^i(t))\left(\mu_{b^i}(t) - \frac{1}{2}\sigma_{b^i}(t)^2\right) + \frac{1}{2}|b^i(t)|\sigma_{b^i}(t)\right] \\ \sigma_\Gamma(t) &= \frac{1}{2}\sum_{i=1}^{N} sgn(b^i(t))\sigma_{b^i}(t) \\ \mu_L(t) &= \frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{\Gamma(t)} - \mu_S(t) + \sigma_S(t)^2 \\ \sigma_L(t) &= \frac{\sigma_\Gamma(t)}{\Gamma(t)} - \sigma_S(t) \end{aligned}$$

Although these expressions are quite complicated, there are a few interesting features to note. First, notice that total debt, $\Gamma(t)$, cannot be written as a geometric Brownian motion, but that it will be positive by definition. Also, notice that the drift in leverage is determined by several terms in $\Gamma(t)$ and drift and diffusion of stock prices. First, you have a term involving $\Gamma(t)$, which is determined by the covariance of stock prices and total debt. Only if debt, its drift, and its diffusion grow proportionally, will the drift in leverage be more or less constant. Finally, notice that diffusion in leverage is decreasing in the diffusion of stock prices. In fact, if $\Gamma(t)$ were to grow faster than $\sigma_\Gamma(t)$, then leverage would move counter-cyclically This will indeed be the case.

Many of these propositions are complex and difficult to grasp in the abstract. They describe the evolution of either an individual agent or the entire economy and involve relative terms and sums. They also reference each other and co-move throughout time. It is because of this complexity that it is easiest to understand how this economy behaves by using a numerical simulation. Armed with Propostions 1 to 5, we can now simulate an economy and study its dynamics over time.

3. Simulation Results and Analysis

In this section, we'll review some simulation results and compare them. For all of the simulations, we will hold the following group of parameters fixed at the given values: $\mu_D = 0.03$, $\sigma_D = 0.06$, and $\rho = 0.01$. These settings correspond to a yearly time discretization. Additionally, for simulating asset prices by Monte Carlo, we need to specify a truncation level $T = 1000$, as well as the number of path iterations $M = 15360$ [2]. Finally, we'll simulate forward 50 periods. We'll begin with simulations using two agents [3] to see how well this model matches previous research. Then we'll move to more agents, focusing on the effects of increasing the number of agents over a fixed support for risk aversion and increasing the spread of the support, holding the number of agents fixed.

Throughout this section we'll focus on two main topics: distributional outcomes and financial market outcomes. In terms of distributional outcomes we will look at consumption shares, wealth distributions, and dominant agents. In terms of financial market outcomes we'll focus on asset prices, dividend yield, interest rates, market price of risk, and risk premia. The model will match better certain variables under some parameterizations and others under other parameterizations. The degrees of freedom are large and the goal of this section is not to perfectly match the real world, but to study in what ways different forms of heterogeneity effect the model.

[2] A multiple of 512 for technical reasons. For a full description of the numerical method, see appendix C. The programs are available on request.

[3] Results for a single agent match the theoretical solution given in appendix A.

3.1. Two Agents

Let's begin with two agents with CRRA parameters $\gamma_1 = 1.0$ and $\gamma_2 = 2.5$. You'll notice in Figure 1 that the agent with the lower risk aversion (the blue line) is dominating the market for consumption. However, an interesting thing to note is that they do not dominate the market for wealth, or at least their position is more volatile and seems to break down as time goes on. The more risk averse agent, although divesting from the risky asset, is buying risk free bonds supplied by the less risk averse agent. So the less risk averse agent is financing the growth in their consumption share through borrowing and, as the rate of growth in asset prices slows (see Figure 3) their leverage tends to rise until their wealth collapses. This also makes their wealth more susceptible to negative shocks. A final interesting feature of Figure 1 is that the variable $\phi(t)$ is never equal to $\xi(t)^2$, which tells us that even the two agent economy could not be matched by a representative agent with time varying risk aversion. This fact is illustrated in the lower pane of Figure 1, which graphs the evolution of the ratio $\frac{\phi(t)}{\xi^2(t)}$. If this ratio is equal to one, then the economy could be modeled by a representative agent, but we see here that it is greater than one. This drives a wedge between interest rates and the market price of risk, increasing the risk premium.

Fig. 1. Distributional outcomes for two agents when $\gamma_1 = 1.0$ (blue) and $\gamma_2 = 2.5$ (red).

Next, consider Figure 2, which describes the financial leverage in the two agent economy. First you'll notice that financial leverage in the whole economy is growing over time. You'll also notice that the dynamics of total leverage in the economy quickly converge to their long run values. Finally, it is very interesting to note that the total financial leverage is counter cyclical, represented by the negative diffusion coefficient in the long run. This is contrary to the assertion in Geanakoplos (2010), that leverage be pro-cyclical That does not mean the two conclusions are opposing, but driven by different forces. In Geanakoplos (2010), a behavioral argument for the effects of heterogeneous beliefs on leverage is given. When there is bad news, credit constraints tighten which causes leverage to fall. Meanwhile, here the change in the level of leverage is driven by agents inter-temporal consumption smoothing motives. They choose their consumption weights without considering asset prices. The market asset price then determines the value of their shares, and the residual is their borrowing. Thus, when the value of their portfolio falls, their borrowing does not necessarily fall. This causes their leverage to rise.

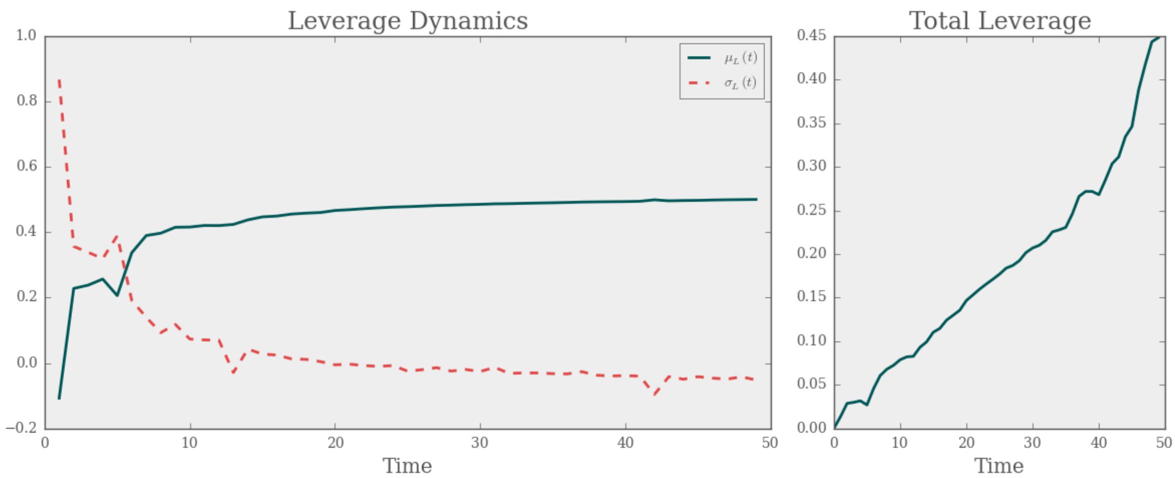

Fig. 2. Financial leverage for two agents when $\gamma_1 = 1.0$ and $\gamma_2 = 2.5$.

Last for this economy, consider Figure 3. Here you'll notice a falling risk free rate and falling market price of risk. Similarly, you'll notice a falling dividend yield, as can be found in the data. This will be a feature of any model with more than one agent, where risk aversion is heterogeneous. Over time, the least risk averse agent is dominating the market for risky assets. You'll notice in Figures 1 to 3 that many of the previously mentioned effects seem to begin to reverse. Towards the end of this simulation, there was a series of large negative shocks that caused the economy to contract. This caused the sudden collapse in the least risk averse agent's wealth and the rise in the dividend yield. In order to better study the effects of heterogeneity, we will now focus on simulations where the risk process realizes at its expectation every period. This will allow us to ignore the effects of shocks, while still studying the dynamics of the economy is a sort of deterministic steady state.

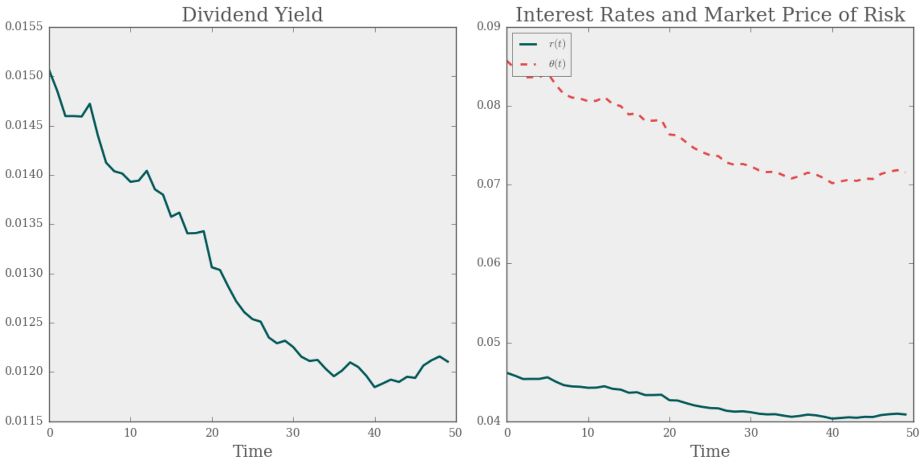

Fig. 3. Financial variables for two agents when $\gamma_1 = 1.0$ and $\gamma_2 = 2.5$.

3.2. Increasing Number of Agents

Let's see how the results change as we increase the number of agents over a fixed support. Throughout this section, the risk aversion parameter of agents will be evenly spaced over the support $[1.0, 5.0]$. As the number of agents increases, the distance between agents shrinks and their initial share in the endowment becomes smaller. This change in the initial share and reduction of concentrations of financial holdings in single points of the distribution will effect on the evolution of consumption shares, wealth processes, as well as financial variables.

First, consider the evolution of consumption shares. Figure 4 shows how individual agents consumption weights evolve over time in each economy. You'll notice that in every case the least risk averse agent dominates the market for consumption in the long run. Also note that the rate of change in their consumption share is greater as the number of agents increases. This implies that they are growing their consumption and asset holdings more quickly when there are more agents. However, the level of their consumption share is much smaller and the most risk averse agent has a consumption share that is falling more slowly as the number of agents increases. So, the lower the initial consumption share and the more evenly spread over the support is the total financial share, the greater is the highest growth rate and the less negative is the lowest growth rate. This is particularly interesting when we compare to the single agent case, where the growth rate in consumption weight would be zero. Moving from one to two agents implies a very large increase in the growth rate of the least risk averse agent's share and a very large decrease in the growth rate of the less risk averse agent's share. However, increasing the number of agents beyond two, which in turn reduces the initial share, increases the growth rate of all agents.

Next, we can examine how increasing the number of agents affects individual wealth. In Figure 5 you see that the increase in the number of agents has an effect of reducing

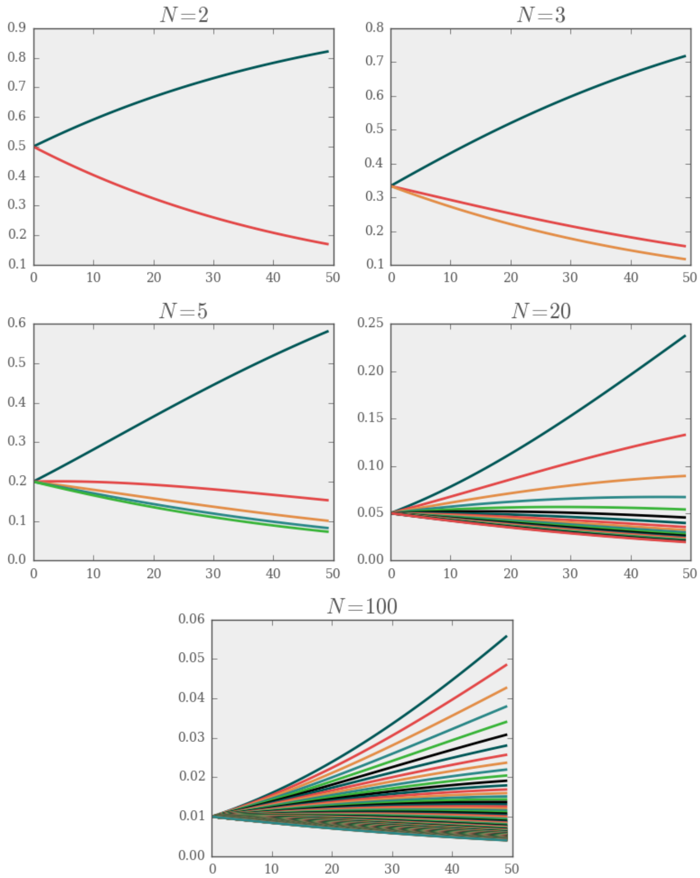

Fig. 4. Evolution of consumption shares for different numbers of agents. Note that the upper most line in each graph corresponds to the agent with the lowest risk aversion, while the lower most line corresponds to the highest risk aversion.

the level of individual wealth. This is caused by two forces. First, the reduction in the agents weight reduces their share of total wealth. Second, the change in the concentration of relative risk aversion reduces asset prices, as will be discussed shortly. Additionally, you'll notice that as the number of agents increases, the wealth of the bottom agent tends to be pulled down into negative territory earlier and earlier. As was just mentioned, the least risk averse agent is able to grow their share of consumption at a faster rate when there are more agents. They do this by borrowing large amounts. In fact, the growth in their borrowing will out-pace the growth in the value of their asset holdings and they will deterministically drift towards a negative wealth. It is important to note that this is true of all of the models, but that this occurs earlier for economies with more agents.

Now, consider how these changes affect the financial market. Figure 6 shows the evolution of $\xi(t)$ and $\phi(t)$, as well as the heterogeneity wedge $\frac{\phi(t)}{\xi^2(t)}$. The addition of more agents has the effect of reducing both $\xi(t)$ and $\phi(t)$, which in turn increases both the risk free rate and the market price of risk. You'll notice first that in all cases the variables are converging towards their long run value 1.0. This is to be expected as the least risk averse agent dominates the market for consumption in the long run. Also, you'll notice

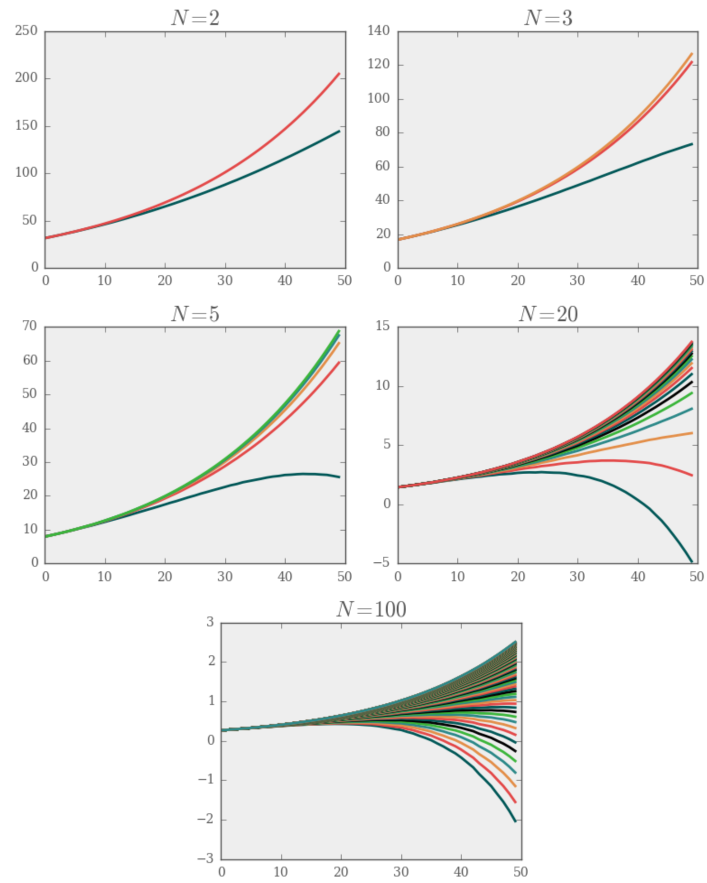

Fig. 5. Evolution of wealth for different numbers of agents. Note that the upper most line in each graph corresponds to the agent with the highest risk aversion, while the lower most line corresponds to the lowest risk aversion. This is the opposite of Figure 4.

that the level of both $\xi(t)$ and $\phi(t)$ is falling as the number of agents rises. The reason that this occurs is that each of these variables is an average of a non-linear transformation of the relative risk aversion, namely the elasticity of inter-temporal substitution $\frac{1}{\gamma}$. So, increasing the number of agents without changing the average risk aversion will have the effect of moving the average elasticity of inter-temporal substitution. These variables determine two averages of the elasticity of inter-temporal substitution and we would expect the lower these values, the higher the risk free rate and market price of risk.

Likewise the heterogeneity wedge moves in a non-linear, non-monotonic way with the number of agents. Moving from two to three agents increases this wedge, while moving from three to five, five to 20, and so forth reduces its value. However, the rate of convergence is affected in the opposite way, as the larger the number of agents the longer the wedge will stay high. These values are quite abstract, but they fully determine the evolution of asset prices and risk premia in this economy. A higher wedge should lead to a higher risk premium.

Given that the risk free rate and the market price of risk are functions of the present realization of $\xi(t)$ and $\phi(t)$, and since these values fall as the number of agents rises

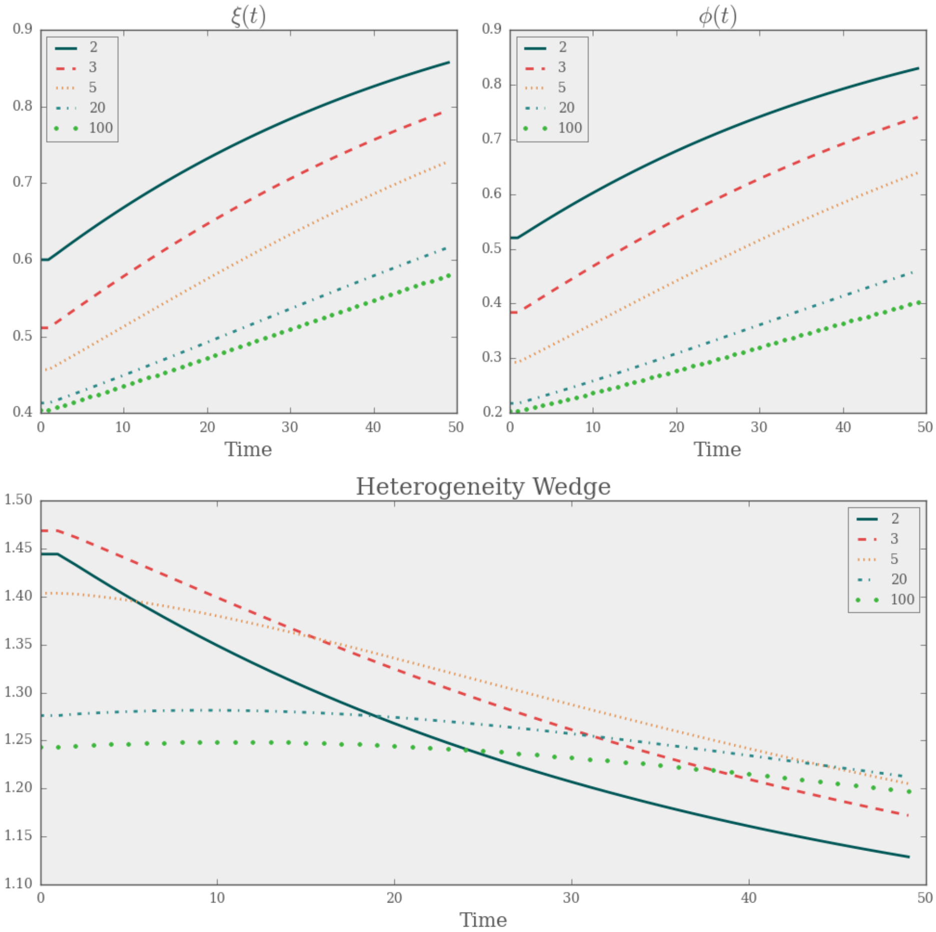

Fig. 6. Weighted averages of elasticity of intertemporal substitution for different numbers of agents, as well as the "heterogeneity wedge" $\frac{\phi(t)}{\xi^2(t)}$.

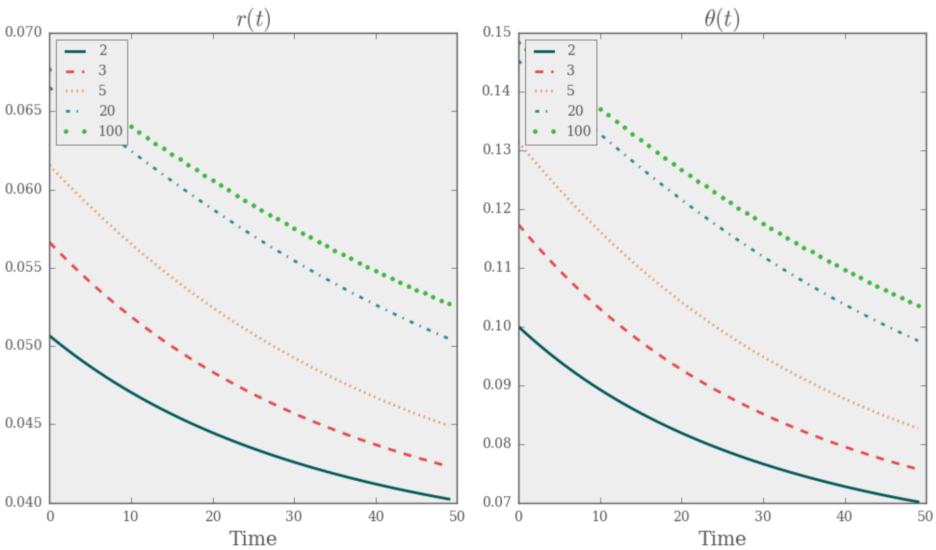

Fig. 7. Interest rate and market price of risk for different numbers of agents.

we expect the risk free rate and market price of risk to rise with the number of agents. Figure 7 shows the level and evolution of $r(t)$ and $\theta(t)$ for different numbers of agents. You'll notice that increasing the number of agents does indeed increase both the risk free rate and the market price of risk. You'll also notice that increasing the number of agents initially increases the speed at which both variables fall, but as the number of agents goes from 5 to 20 and from 20 to 50, this speed falls. As it takes a longer period of time for a single agent to dominate the economy, prevailing interest rate and market price of risk remain high longer.

Next, consider the combined effect of all that we've discussed on asset prices. Recall that asset prices are a sum of the present discounted value of dividends, where the stochastic discount factor is determined by $\xi(t)$ and $\phi(t)$. If these two variables are low, they cause the risk free rate and market price of risk to be high, as we've just seen, which means the discount is greater. By a larger discount I mean that the stochastic discount factor falls more quickly, discounting more future dividends. So, we would expect the price of assets to be lower in an economy with higher average risk aversion and since increasing the number of agents increases this average, increasing the number of agents should reduce asset prices. This is indeed the case, as can be seen in Figure 8, where economies with more agents have a higher dividend yield. Additionally, you'll notice that the dividend yield is falling for every simulation. This matches the real world, in which the market dividend yield has been trending downward since the beginning of financial liberalization of the 1980s

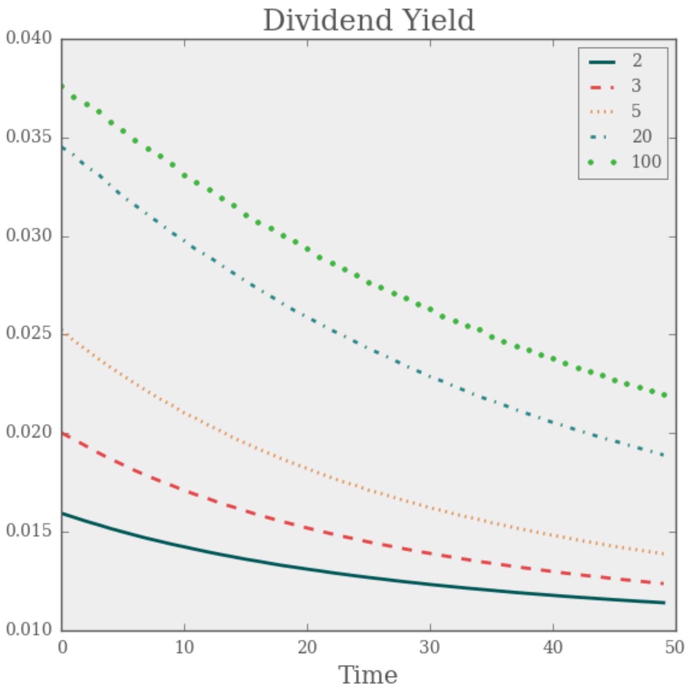

Fig. 8. Dividend yield, $\frac{D(t)}{S(t)}$, for different numbers of agents.

In this economy, expected excess returns are identical to the equity risk premium and the addition of more agents does in fact increase this equity risk premium. We will examine each component of $ERP(t) = \mu_S(t) + \frac{D(t)}{P(t)} - r(t)$ individually (Table 2 contains simulation results) and try to come to a justification for this claim. First, the dividend-yield is falling in an economy with more than one agent. This implies that the price of the risky asset is growing more quickly than the dividend process. Thus, the expected growth rate in prices must be greater than the expected growth rate in dividends. Since in a representative agent economy, $\mu_S(t) = \mu_D$ it has to be that $\mu_S(t)$ is strictly larger in an economy with more than one agent than in an economy with a single agent. The magnitude of the difference between $\mu_S(t)$ and μ_D is determined by how quickly $S(t)$ is converging towards the price that would prevail in a representative agent economy populated by the agent with the lowest risk aversion. You'll notice in Figure 8 that the slope of the dividend yield as a function of time seems to be increasing as the number of agents rises. However, the difference between 20 and 100 agents is not so clear and could even move in the opposite direction. This non-linearity/non-monotonicity in the effect of increasing the number of agents is similar to that discussed in regard to the heterogeneity wedge.

Number of Agents	$\mu_S(0)$	$\frac{D(0)}{S(0)}$	$r(0)$	$ERP(0)$
2	0.041	0.016	0.050	0.007
3	0.045	0.020	0.057	0.008
5	0.049	0.025	0.062	0.012
20	0.043	0.035	0.067	0.010
100	0.038	0.038	0.068	0.008
Number of Agents	$\mu_S(50)$	$\frac{D(50)}{S(50)}$	$r(50)$	$ERP(50)$
2	0.032	0.011	0.040	0.003
3	0.033	0.012	0.042	0.003
5	0.035	0.014	0.045	0.004
20	0.037	0.019	0.050	0.006
100	0.039	0.022	0.052	0.008
Number of Agents	$\bar{\mu_S}$	$\frac{D}{S}$	$\bar{r}$	$E\bar{R}P$
2	0.035	0.013	0.044	0.004
3	0.039	0.015	0.048	0.006
5	0.041	0.018	0.052	0.007
20	0.041	0.025	0.058	0.008
100	0.040	0.029	0.059	0.010

Table 1: This table reports the initial period, terminal period, and the average (denoted by a bar, e.g. $\bar{x}$) outcomes from simulations spanning 50 periods, using different numbers of agents, and with a zero realization for the risk process every period ($dW(t) = 0 \ \forall \ t$) to abstract away from the influence of random shocks. The RRA parameter in these simulations are uniformly distributed over $[1.0, 5.0]$.

Next, the dividend yield is increasing in the number of agents. Recall that the price of the risky stock is the discounted sum of future dividends. Since the discount factor converges toward zero at a rate determined by the interest rate and the market price of risk, if these remain high then the discount is larger, reducing asset prices. Again, with more agents the interest rate converges more slowly, as does the market price of risk. Since the interest rate and market price of risk remain high for a longer period of time, the price of risky stocks is lower when there are more agents, and in turn the dividend-yield higher. So, a higher growth rate $mu_S(t)$ and a higher dividend-yield $\frac{D(t)}{S(t)}$ increase the equity risk premium.

Finally, the risk free rate is increasing in the number of agents. This has the effect of pushing the equity risk premium in the opposite direction. The question of which effect will dominate is not clear. Table 1 gives the values of $\mu_S(t)$, $\frac{D(t)}{S(t)}$, and $r(t)$ in the initial period, the terminal period, and on average for several simulations. Indeed the risk premium is increasing in the long run and on average, but in the initial period the relationship is non-monotonic. Moving from 2 to 5 agents we see a doubling of the risk premium in the first period, while the increase all but disappears in moving from 5 to 100 agents. First, the dividend yield is monotonically increasing in the number of agents, although the rate of change is falling. Second, as the number of agents increases, the economy is slower to converge to its long run level and so the expected growth rate in stock prices begins to fall, partially or completely canceling out the increase in dividend yields. Finally, the interest rate is monotonically increasing in the number of agents, pulling down the risk premium. This is a manifestation of the initial fall in the heterogeneity wedge seen in Figure 6. However, the fact that the equity risk premium remains high in the long run is caused by the slow decay of this wedge seen in the same figure. We can conclude that on average the equity risk premium is higher for an economy with more agents.

The last thing to consider is the evolution of leverage. Figure 9 shows the drift and level of leverage for the different economies. For this short simulation of only 50 periods there is little divergence between the different economies and the drift of leverage seems very close across models. However, you'll notice that the level seems to be diverging over time. This is because in the many agent case, as the economy begins to converge there are more and more agents whose risk aversion is above the weighted average and are thus willing to purchase bonds. So, the supply of bonds is rising in the economy and the less risk averse agents are more than happy to demand those bonds in order to purchase the risky asset. In this way when there are more agents total leverage tends to rise more quickly.

Taking into account only total leverage neglects the individual outcomes. Figure 10 plots the average over agents of total financial leverage and total debt for each economy. In economies with more agents the average leverage per individual is very small, as is the average debt. However, this calculation divides total leverage by the total number of

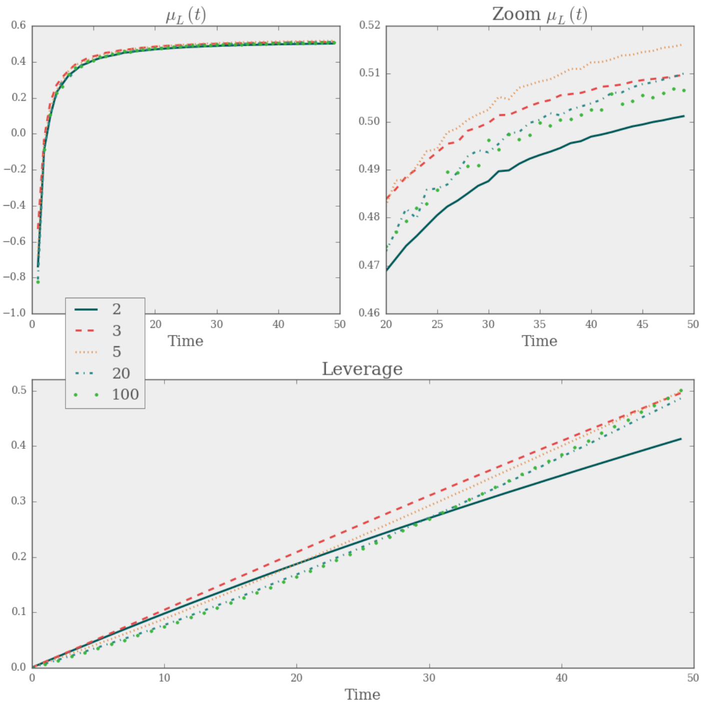

Fig. 9. Leverage and its drift for different numbers of agents.

agents, even though the number of agents borrowing is much lower. For instance, in the three agent economy there is only one borrower, so the average leverage, conditional on being leveraged, would be three times the number given in the graph. In the 20 agent economy there are 5 borrowers, so the number is four times as large as that given in the graph. So, if we were to graph the average leverage, conditional on being a borrower, the effect of higher leverage in economies with fewer agents would be smaller and the level of leverage would be much higher. That being said, we can conclude that an economy with more agents has a lower level of individual leverage and borrowing.

To summarize, economies with more agents, and thus a lower individual share of total consumption initially, exhibit slower convergence towards a long run where one agent dominates, lower individual wealth, a higher interest rate, a higher market price for risk, lower asset prices, higher expected growth in asset prices, a higher equity risk premium, greater total leverage, and lower individual leverage.

3.3. Increasing Spread

We can also study the effects of widening the support for risk aversion. To accomplish this we'll look at an economy populated by only two agents and centered around a widely accepted value for the average relative risk aversion in the economy: 3.0. Then we'll look at outcomes for the model for different values for these agents relative risk aversion, moving them progressively further apart. This essentially increases the amount of disagreement between the two agents, in that if they were alone in the economy their re-

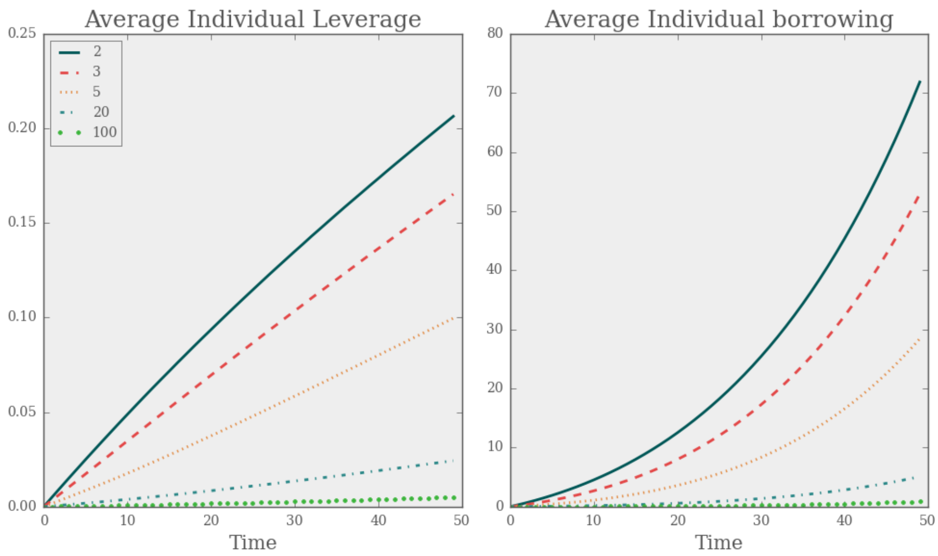

Fig. 10. Average individual leverage for different numbers of agents.

spective prices and interest rates would be more and more different the wider the spread. As we've already seen, the outcome of the model is determined by weighted averages of the EIS, or $\frac{1}{\gamma}$. Since this is a non-linear, convex transformation, the increase in the spread of the support will cause the initial values of $\xi(t)$ and $\phi(t)$ to fall, the combined effect of which will change the initial level of financial variables and the speed of convergence. This change will have a large effect on the risk premium and distributional outcomes in the model.

First, consider how the evolution of consumption shares changes as the spread between agents grows. Figure 11 illustrates how increasing the spread between the two agents causes their consumption shares to diverge more quickly. Increasing the spread causes each individual's preferences to be further from the average, increasing their level of disagreement. This causes the more risk averse agent to desire to divest and the less risk averse agent to desire to invest.

Next, we can think about how the change in the spread affects the evolution of individual wealth. Figure 12 shows individual wealth for each economy. You'll notice that the level of individual wealth is increasing as the spread grows. You'll also notice that the rates of change in the wealth of the two agents seem to move in opposite directions. Increasing the spread in their risk aversion from 0.2 to 1.0 or 2.0 seems to increase the growth rate in the most risk averse agents wealth and decrease that of the least risk averse agent's wealth. However, increasing the spread above 2.0 seems to reverse this effect. In the first case, the economy is converging slowly enough that the growth in asset prices cannot out-pace the growth of the less risk averse agents borrowing. However, as the spread increases the economy converges more quickly and the rate of growth in asset prices (as you'll see shortly in Table 2) is enough to out-pace their borrowing.

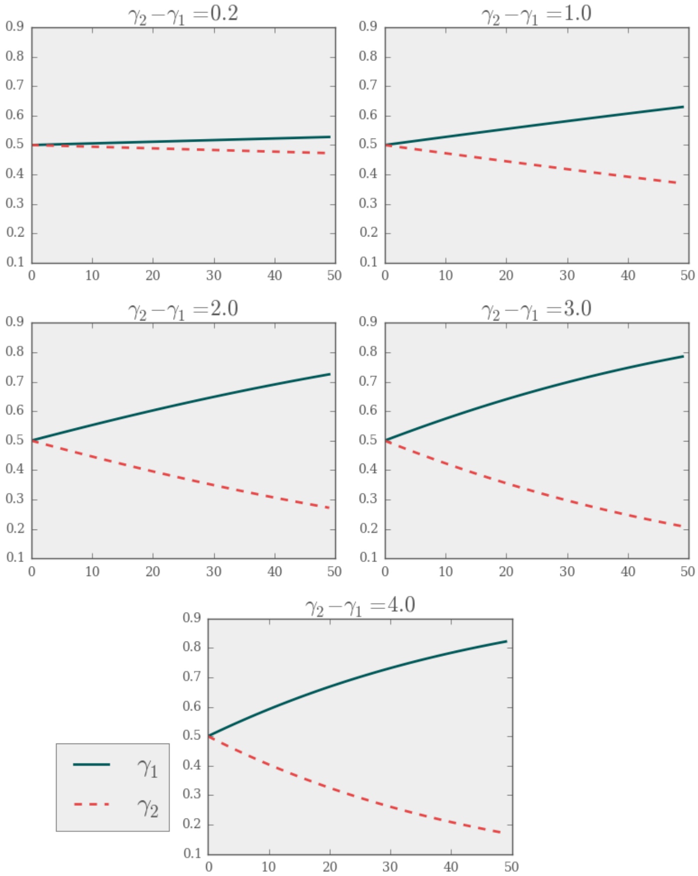

Fig. 11. Evolution of consumption shares for different values of $\gamma_2 - \gamma_1$.

Now, consider how the dynamics of consumption shares affect the values of $\xi(t)$ and $\phi(t)$, represented in Figure 13. The effect is opposite to that of increasing the number of agents, as both $\xi(t)$ and $\phi(t)$ rise as the spread increases. Also, the effect on the heterogeneity wedge is opposite, as increasing the spread between the two agents causes the ratio $\frac{\phi(t)}{\xi^2(t)}$ to rise. The direct effect of this is to reduce interest rates and the market price of risk. In Figure 14 you see that increasing the spread between the two agents has the effect of simultaneously reducing both the interest rate and market price of risk, as well as increasing their rate of change. The larger difference in the two agents' risk aversion parameters, the higher the values of $\xi(t)$ and $\phi(t)$. So the increase in the spread has the effect of pushing down the weighted average of risk aversion in the economy. This effect will cause increasing the spread of the distribution to have opposing effects to those from increasing the number of agents. Also, the faster rate of convergence in consumption increases the rate of change in $r(t)$ and $\theta(t)$. Finally, the reduction in the risk aversion of the least risk averse agent reduces the long run value of both the interest rate a market price of risk.

Next, notice the effect on the heterogeneity wedge in Figure 13. As this heterogeneity wedge rises, the difference in the interest rate and the market price of risk grow, in that

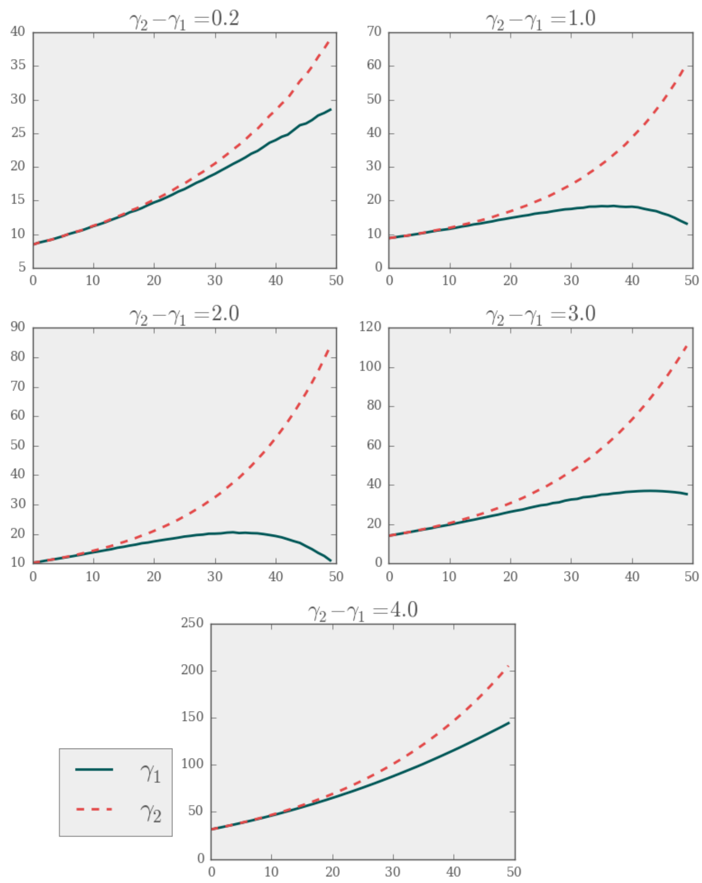

Fig. 12. Evolution of individual wealth for different values of $\gamma_2 - \gamma_1$.

the values one would obtain using a representative agent become less and less accurate. One could match either the market price of risk or the interest rate, but not both. Since we are widening the spread, pushing down the average risk aversion, the effect of this wedge is to hold up interest rates while reducing the market price of risk, reducing the equity risk premium.

Using this information, we can study the evolution of asset prices. Figure 15 shows how the dividend yield evolves for each economy. The most striking feature of this graph is that as the spread increases the level of the dividend yield is much lower. This is because as the spread grows, the interest rate and market price of risk fall, increasing the value of the stochastic discount factor each period and in turn increasing asset prices. Second, you'll notice that the dividend yield is falling over time for economies with a larger spread. This is again matching more closely the real world, as in Figure 8. However, increasing the spread between agents has an effect on the level of the dividend yield that moves in the opposite direction to that of increasing the number of agents.

Indeed, the increase in the spread also has the opposite effect of the increase in the number of agents on the equity risk premium, while increasing the expected growth rate in stock prices and decreasing the dividend yield. Table 2 shows the outcomes for the

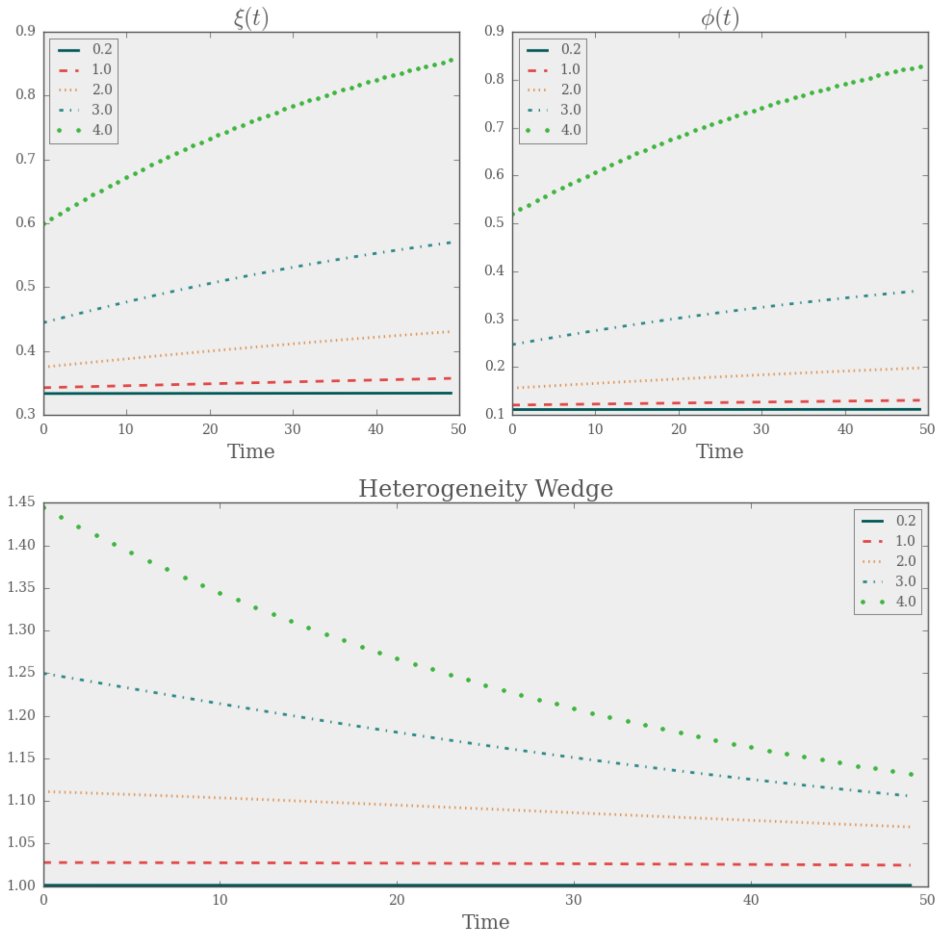

Fig. 13. Weighted averages of elasticity of intertemporal substitution for different values of $\gamma_2 - \gamma_1$, as well as the "heterogeneity wedge" $\frac{\phi(t)}{\xi^2(t)}$.

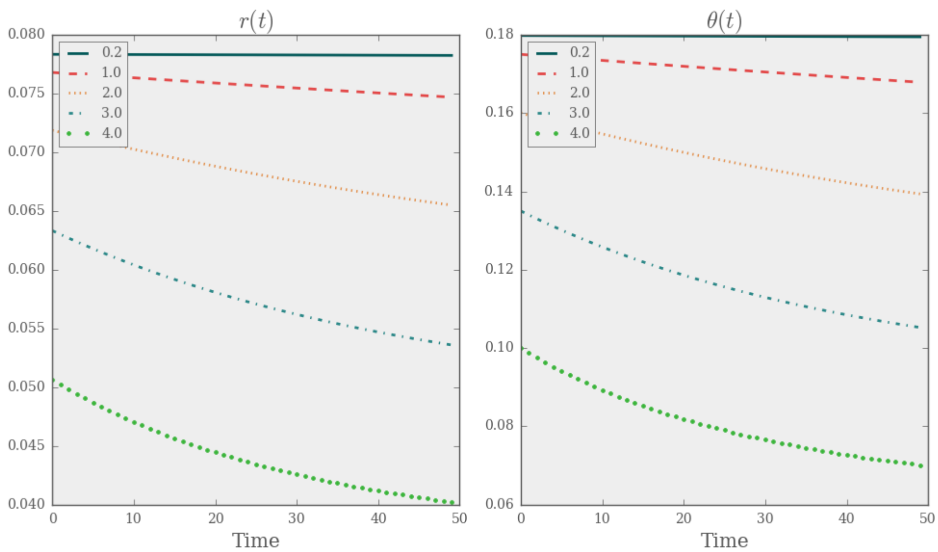

Fig. 14. Interest rate and market price of risk for different values of $\gamma_2 - \gamma_1$.

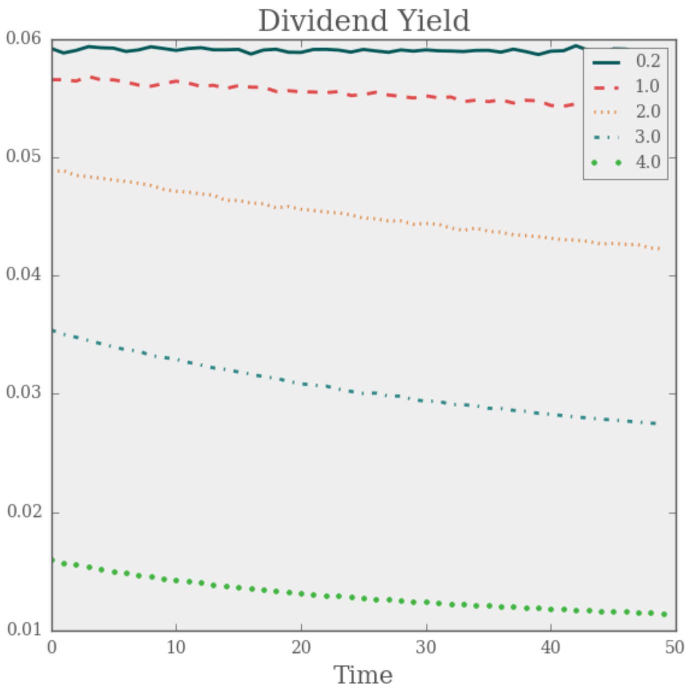

Fig. 15. Evolution dividend yield for different values of $\gamma_2 - \gamma_1$.

$\gamma_2 - \gamma_1$	$\mu_S(0)$	$\frac{D(0)}{S(0)}$	$r(0)$	$ERP(0)$
0.2	0.035	0.059	0.078	0.016
1.0	0.028	0.057	0.077	0.008
2.0	0.027	0.049	0.072	0.004
3.0	0.039	0.035	0.063	0.011
4.0	0.042	0.016	0.051	0.007
$\gamma_2 - \gamma_1$	$\mu_S(50)$	$\frac{D(50)}{S(50)}$	$r(50)$	$ERP(50)$
0.2	0.031	0.059	0.078	0.012
1.0	0.028	0.054	0.075	0.007
2.0	0.032	0.042	0.065	0.009
4.0	0.034	0.027	0.053	0.008
5.0	0.032	0.011	0.040	0.003
$\gamma_2 - \gamma_1$	$\bar{\mu_S}$	$\frac{D}{S}$	$\bar{r}$	$E\bar{R}P$
0.2	0.030	0.059	0.078	0.010
1.0	0.029	0.055	0.076	0.008
2.0	0.032	0.045	0.068	0.009
3.0	0.034	0.031	0.058	0.007
4.0	0.036	0.013	0.044	0.005

Table 2: This table reports the initial period, terminal period, and the average (denoted by a bar, e.g. $\bar{x}$) outcomes from simulations spanning 50 periods, using two agents, and with a zero realization for the risk process every period ($dW(t) = 0 \; \forall \; t$) to abstract away from the influence of random shocks. The RRA parameter in these simulations are progressively further apart, implying a greater and greater disagreement between the two agents.

expected growth rate in stock prices, the dividend yield, and the interest rate. As we increase the spread, the two agents disagree to a larger degree and so the more risk averse agent will divest more quickly from their risky asset position, causing the price to rise more quickly and to a higher level, given the fact that the long run dominant agent is less risk averse. Simultaneously the average EIS will rise, pushing up the level of asset prices and reducing the dividend yield. This rise in the average EIS also reduces interest rates, but the fall in the dividend yield outweighs the rise in expected returns minus the interest rate. The overall effect is that an increase in the spread of the distribution reduces the equity risk premium.

Finally, consider the amount of leverage in the different economies, represented in Figures 16 and 17 show the evolution of total leverage, the drift of total leverage, average individual leverage, and average individual debt. You'll notice that the level differs greatly, as do the long run growth rates. The same disagreement in asset prices that drove consumption choices is at work here. Increasing the spread between the two agents causes their level of disagreement to be greater and the rate of change in their consumption weights to be larger. They finance this change through debt. However, a higher spread implies higher asset prices. For a large enough spread the higher asset prices reduce the level of long run leverage.

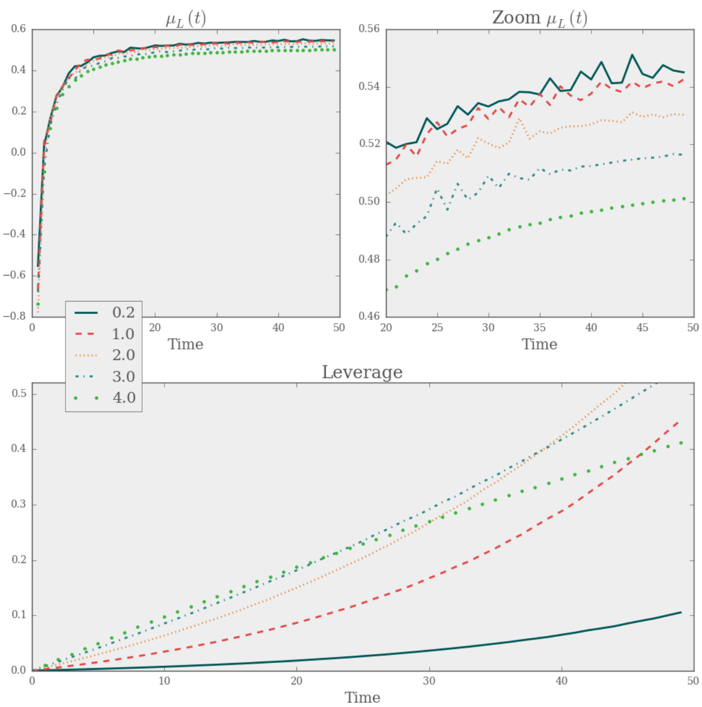

Fig. 16. Expected growth rate and level of leverage for different values of $\gamma_2 - \gamma_1$.

To summarize, economies with two agents whose risk aversion parameters are farther apart exhibit faster convergence towards a long run where one agent dominates, higher individual wealth, a lower interest rate, a lower market price for risk, higher asset prices,

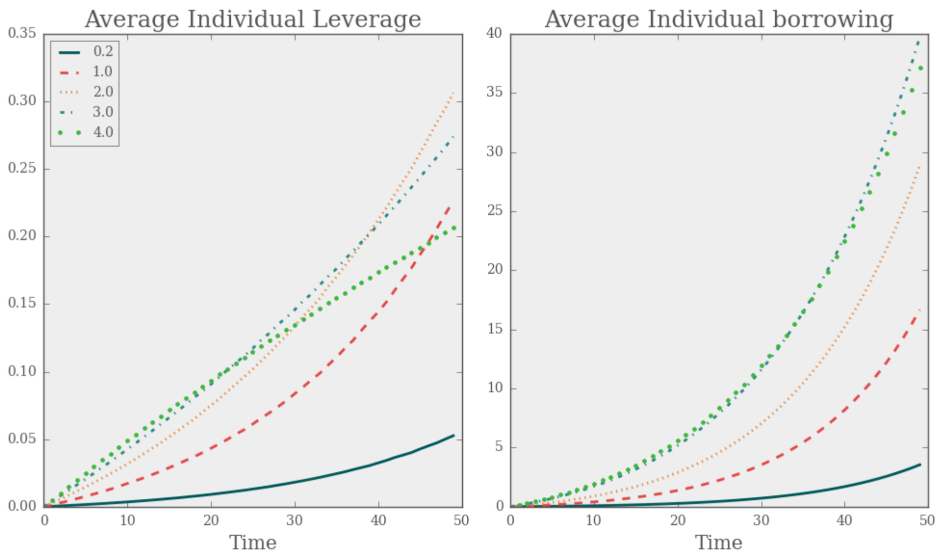

Fig. 17. Average individual leverage for different values of $\gamma_2 - \gamma_1$.

lower expected growth in asset prices, and a lower equity risk premium. However, the effect on total leverage and individual leverage is ambiguous.

4. Conclusion

As we've seen, the addition of more agents to a model with heterogeneous preferences has a substantial effect on model predictions. Similarly, increasing the degree to which agents differ in their risk preferences has an opposing effect, in many respects. This could be interpreted in several ways, but I will attempt to do so in an inequality sense. Increasing the number of agents can be thought of as decreasing inequality, while increasing the spread between two agents can be thought of as increasing inequality. As the number of agents increases over a given spread, the share each agent holds falls and the distance between each agent in terms of risk preferences falls. Increasing the spread between two agents has exactly the opposite effect, increasing the distance between preferences. In this sense, the one is decreasing inequality while the other is increasing it.

Increasing inequality in this model leads to several phenomena for further study. First, inequality grows more quickly in an unequal society. As individual agents command larger shares of financial markets and as their preferences differ to a greater degree with their compatriots, they accumulate financial shares at a larger rate. Second, more unequal societies see higher levels of asset prices. As one individual begins to dominate the economy, financial shares will accumulate to the least risk averse agent, whose price will prevail in the market. Since this agent has the highest individual asset price, the market price will be high. Third, excess returns in the market will be lower in a more unequal society. When the least risk averse agent dominates the market, expected growth in asset prices and dividend yield fall, but more quickly than interest rates. This has the effect of reducing the equity risk premium. Finally, in an economy populated by agents who differ in their risk preferences, the most risk averse agent will dominate the market for wealth and all other agents will eventually see their debt-financed consumption turn their financial wealth negative. However, in a more unequal society, this will occur much later.

This last point can be interpreted in two ways. Consider a world where when an agent's financial wealth becomes negative they default and their assets are redistributed to their lenders. On first glance, one would say that in a more equal society, default will occur more often as agents are unable to finance their consumption growth on the crowded market. This is true, but what is more important is the size of the default. In a more unequal society, when an agent does eventually default, which they surely will, the amount of borrowing they hold will be larger, and the collapse in asset prices larger, as their shares are transferred to more risk averse agents. Thus, in a more unequal society financial default of individual agents would have greater spillover effects into the larger economy. If this is true, it would imply that financial crises in societies with greater inequality are more severe.

This paper represents only a first step towards, hopefully, very enlightening aspects of heterogeneity and complexity in finance. Here, we've seen that simply increasing the number of agents over the same support of a distribution of relative risk aversion or the spread in the support can have effects on the amount and volatility of leverage in an economy, the rate of change in the dividend price ratio, and can cause a glut of lending. Further research on this topic could include an OLG framework that would allow for finite horizons, financial frictions that limit borrowing, or allowing for default in borrowing.

Appendix A. The Representative Agent Solution

This appendix contains (without proof) the solution to the representative agent economy, i.e. the case where $N = 1$.

$$
\begin{aligned}
r(t) &= \rho + \gamma\mu_D - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2 \\
\theta(t) &= \sigma_D\gamma \\
\frac{S(t)}{D(t)} &= \frac{1}{\rho + \mu_D(\gamma-1) - \frac{1}{2}\sigma^2\gamma(\gamma-1)} \\
\mu_S(t) &= \mu_D \\
\sigma_S(t) &= \sigma_D
\end{aligned}
\tag{18}
$$

Appendix B. Proofs

Proof of Proposition 1. Taking ratios of consumption first order conditions for two arbitrary agents, i and j we find

$$
\frac{c^i(t)}{c^j(t)} = \Lambda_j^{\frac{1}{\gamma_j}} \Lambda_i^{\frac{-1}{\gamma_i}} \left(H_0(t)e^{\rho t}\right)^{\frac{1}{\gamma_j} - \frac{1}{\gamma_i}}
$$

To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption

$$
\begin{aligned}
& \sum_{j=1}^{N} c^j(t) = D(t) \\
\Leftrightarrow \quad & \frac{\sum_{j=1}^{N} c^j(t)}{c^i(t)} = \frac{D(t)}{c^i(t)} \\
\Leftrightarrow \quad & c^i(t) = \frac{c^i(t)}{\sum_{j=1}^{N} c^j(t)} D(t) \\
\Leftrightarrow \quad & c^i(t) = \left(\frac{\left(e^{\rho t}\Lambda_i(t)H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N} \left(e^{\rho t}\Lambda_j(t)H_0(t)\right)^{\frac{-1}{\gamma_j}}} \right) D(t) \\
\Leftrightarrow \quad & c^i(t) = \omega^i(t)D(t)
\end{aligned}
$$

$\square$

Proof of Lemma 1. Modeling consumption as a geometric Brownian motion implies that for every agent i the consumption process can be described by the stochastic differential equation

$$\frac{dc^i(t)}{c^i(t)} = \mu_{c^i}(t)dt + \sigma_{c^i}(t)dW(t) \tag{19}$$

Armed with this knowledge, take the first order condition for an arbitrary agent i's maximization problem, solve for $H_0(s)$, and apply Itô's lemma:

$$\begin{aligned} H_0(t) &= \frac{1}{\Lambda_i} c^i(t)^{-\gamma_i} e^{-\rho t} \\ \Rightarrow \frac{dH_0(t)}{H_0(t)} &= \left(-\rho - \gamma\mu_{c^i}(t) + \gamma(1+\gamma)\frac{\sigma_{c^i}(t)^2}{2}\right) dt - \left(\gamma_i \sigma_{c^i}(t)\right) dW(t) \end{aligned}$$

Now, match coefficients to those in (7) to find

$$\begin{aligned} r(t) &= \rho + \gamma_i \mu_{c^i}(t) - \gamma_i(1+\gamma_i)\frac{\sigma_{c^i}(t)^2}{2} \\ \theta(t) &= \gamma_i \sigma_{c^i}(t) \end{aligned}$$

Solving for μ_{c^i} and σ_{c^i} gives

$$\begin{aligned} \mu_{c^i}(t) &= \frac{r(t) - \rho}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2} \frac{\theta(t)^2}{2} \\ \sigma_{c^i}(t) &= \frac{\theta(t)}{\gamma_i} \end{aligned}$$

□

Proof of Proposition 2. Recall the definition of consumption dynamics in (19) and the market clearing condition for consumption in (4). Apply Itô's lemma to the market clearing condition:

$$\begin{aligned} &\sum_{i=1}^{N} c^i(t) = D(t) \Rightarrow \sum_{i=1}^{N} dc^i(t) = dD(t) \\ \Leftrightarrow &\sum_{i=1}^{N} \left(c^i(t)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right) = D(t)\mu_D dt + D(t)\sigma_D dW(t) \\ \Leftrightarrow &\frac{\sum_{i=1}^{N} \left(c^i(s)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right)}{\sum_{i=1}^{N} c^i(t)} = \mu_D dt + \sigma_D dW(t) \\ \Leftrightarrow &\sum_{i=1}^{N} \omega^i(t)\mu_{c^i}(t)dt + \sum_{i=1}^{N} \omega^i(t)\sigma_{c^i}(t)dW(t) = \mu_D dt + \sigma_D dW(t) \end{aligned}$$

By matching coefficients we find

$$\mu_D = \sum_{i=1}^{N} \omega^i(t)\mu_{c^i}(t)$$
$$\sigma_D = \sum_{i=1}^{N} \omega^i(t)\sigma_{c^i}(t)$$

Now use Lemma 1 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk to find

$$\theta(t) = \frac{\sigma_D}{\xi(t)}$$
$$r(t) = \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t)+\phi(t)}{\xi(t)^3}\sigma_D^2$$

where

$$\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$$
$$\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$$

□

Proof of Proposition 3. Assume that consumption weights follow a geometric Brownian motion given by

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t) \tag{20}$$

Recall the definition of consumption weights in (10) and gather terms:

$$\omega^i(t) = \frac{\left(\Lambda^i e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda_j e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_j}}}$$
$$\Leftrightarrow \omega^i(t) = \left[\sum_{j=1}^{N} \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_0(t)\right)^{\frac{1}{\gamma_i}-\frac{1}{\gamma_j}}\right]^{-1} \tag{21}$$

Recall the definition of Itô's lemma, where $\omega^i(t)$ is a function of $H_0(t)$ and t:

$$d\omega_i(t) = \frac{\partial \omega^i(t)}{\partial t}dt + \frac{\partial \omega^i(t)}{\partial H_0(t)}dH_0(t) + \frac{1}{2}\frac{\partial^2 \omega^i(t)}{\partial H_0(t)^2}(dH_0(t))^2$$

Substituting for $dH_0(t)$ by (7) and using the Itô box calculus to see that $(dH_0(t))^2 =$

$H_0(t)^2\theta(t)^2 dt$, we see that

$$\frac{d\omega^i(t)}{\omega^i(t)} = \frac{1}{\omega^i(t)}\left(\frac{\partial\omega^i(t)}{\partial t} - r(t)H_0(t)\frac{\partial\omega^i(t)}{\partial H_0(t)} + H_0(t)^2\theta(t)^2\frac{1}{2}\frac{\partial^2\omega^i(t)}{\partial H_0(t)^2}\right)dt$$
$$- \theta(t)\frac{1}{\omega^i(t)}\frac{\partial\omega^i(t)}{\partial H_0(t)}dW(t)$$

Matching coefficients with those in (20) it is clear that

$$\mu_{\omega^i} = \frac{1}{\omega^i(t)}\left(\frac{\partial\omega^i(t)}{\partial t} - r(t)H_0(t)\frac{\partial\omega^i(t)}{\partial H_0(t)} + H_0(t)^2\theta(t)^2\frac{1}{2}\frac{\partial^2\omega^i(t)}{\partial H_0(t)^2}\right)$$
$$\sigma_{\omega^i} = -\theta(t)\frac{1}{\omega^i(t)}\frac{\partial\omega^i(t)}{\partial H_0(t)}$$

Differentiating the expression in (21), carrying out some painful algebra, and simplifying gives

$$\mu_{\omega^i}(t) = (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \theta(t)^2\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right]$$
$$\sigma_{\omega^i}(t) = \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right)$$

□

Proof of Proposition 4. Following a trick in Gârleanu and Panageas (2008), we can arrive at an expression for asset prices. Take a straight forward application of Itô's lemma to the time t present value of time u wealth:

$$\begin{aligned} d(H_0(u)X^i(u)) &= X^i(u)dH_0(u) + H_0(u)dX^i(u) + dH_0(u)dX^i(u) \\ &= X^i(u)\left(-r(u)H_0(u) - \theta(u)H_0(u)dW(u)\right) \\ &\quad + H_0(u)\left[\left(r(u)X^i(u) + a^i(u)S(u)\left(\mu_S(u) + \frac{D(u)}{S(u)} - r(u)\right) - c^i(u)\right)du\right. \\ &\quad \left. a^i(u)S(u)\sigma_S(u)dW(u)\right] - \theta(u)H_0(u)\omega^i(u)\sigma_S(u)S(u)du \end{aligned}$$

Now, notice that in this economy agents asset holdings are simply their consumption weight (ie $c^i(s) = \omega^i(s)D(s)$ and $a^i(s) = \omega^i(s)$) and that $\sigma_S(t)\theta(t) = \mu_S(t) + \frac{D(t)}{S(t)} - r(t)$ by (6). This implies that the above expression simplifies to

$$d(H_0(s)X^i(s)) = -H_0(u)\omega^i(u)D(u)du + H_0(u)[\omega^i(u)\sigma_S(u)S(u) - X^i(u)\theta(u)]dW(u)$$

By the definition of the Itô differential this is equivalent to

$$\lim_{u\to\infty} H_0(u)X^i(u) - H_0(t)X^i(t) = -\int_t^\infty H_0(u)\omega^i(u)D(u)du \\ + \int_t^\infty H_0(u)[\omega^i(u)\sigma_S(u)S(u) - X^i(u)\theta(u)]dW(u)$$

If we take expectations, then the first term on the left hand side is zero by a transversality condition on the present value of wealth. Also, notice that the Brownian integral on the right hand side is zero in expectation by the martingale property (Oksendal (1992)). So we can write

$$-H_0(t)X^i(t) = -\mathbb{E}_t \int_t^\infty H_0(u)\omega^i(u)D(u)du$$

Finally, we arrive at an expression for wealth today

$$X^i(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\omega^i(u)D(u)du$$

Now take the market clearing condition for wealth and substitute this new formula

$$\begin{aligned} S(t) &= \sum_{i=1}^N X^i(t) \\ &= \sum_{i=1}^N \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\omega^i(u)D(u)du \\ &= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\left(\sum_{i=1}^N \omega^i(u)\right) D(u)du \\ &= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u)du \end{aligned}$$

□

Proof of Lemma 2. Individual savings is determined by wealth and asset holdings: $b^i(t) = X^i(t) - \omega(t)S(t)$. By an application of Itô's product rule ($d(XY) = YdX + XdY + dXdY$) and by (3), (2), and Proposition 2, we can arrive at an expression for the dynamics of individual savings:

$$
\begin{aligned}
db^i(t) &= dX^i(t) - S(t)d\omega^i(t) - \omega^i(t)dS(t) - d\omega^i(t)dS(t) \\
&= \left[r(t)X^i(t) + \omega^i(t)S(t)(\mu_S(t) - r(t))\right] dt + \omega^i(t)\sigma_S(t)S(t)dW(t) \\
&- S(t)\omega^i(t)(\mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t) - S(t)\omega^i(t)(\mu_S(t)dt + \sigma_S(t)dW(t)) \\
&- \omega^i(t)S(t)\sigma_{\omega^i}(t)\sigma_S(t)dt
\end{aligned}
$$

Substituting the definition for $b^i(t)$ and simplifying gives

$$
db^i(t) = [r(t)b^i(t) - \omega^i(t)S(t)(\mu_{\omega^i}(t) + \sigma_S(t)\sigma_{\omega^i}(t))]dt - \sigma_{\omega^i}(t)S(t)\omega^i(t)dW(t)
$$

Finally, matching coefficients gives the result:

$$
db^i(t) = \mu_{b^i}(t)dt + \sigma_{b^i}(t)dW(t)
$$

where

$$
\begin{aligned}
\mu_{b^i}(t) &= r(t)b^i(t) - \omega^i(t)S(t)(\mu_{\omega^i}(t) + \sigma_S(t)\sigma_{\omega^i}(t)) \\
\sigma_{b^i}(t) &= -\sigma_{\omega^i}(t)\omega^i(t)S(t)
\end{aligned}
$$

□

Proof of Proposition 5. For this proof we need an intermediate result about Itô's lemma and absolute values. If we think of $|X(t)| = (X(t)^2)^{\frac{1}{2}} = f(X(t))$ where $dX(t) = \mu_X(t)dt + \sigma_X(t)dW(t)$, then we can use Itô's lemma to show that

$$
d|X(t)| = \left(\frac{X(t)}{|X(t)|}\mu_X(t) + \frac{1}{2}\frac{|X(t)|^2 - X(t)}{|X(t)|}\sigma_X(t)^2\right) dt + \sigma_X(t)\frac{X(t)}{|X(t)|}dW(t) \tag{22}
$$

Now, we can use (22), along with Lemma 2 to find the dynamics of total debt in the economy:

$$
\begin{aligned}
d\Gamma(t) &= d\left(\frac{1}{2}\sum_{i=1}^{N}|b^i(t)|\right) \\
&= \frac{1}{2}\sum_{i=1}^{N} d|b^i(t)| \\
&= \frac{1}{2}\sum_{i=1}^{N}\left[\left(\frac{b^i(t)}{|b^i(t)|}\mu_b^i(t) + \frac{1}{2}\frac{|b^i(t)|^2 - b^i(t)}{|b^i(t)|}\sigma_b^i(t)^2\right) dt + \sigma_b^i(t)\frac{b^i(t)}{|b^i(t)|}dW(t)\right]
\end{aligned}
$$

Notice that $\frac{b^i(t)}{|b^i(t)|} = sgn(b^i(t))$...

$$d\Gamma(t) = \frac{1}{2}\sum_{i=1}^{N}\left[sgn(b^i(t) + \frac{1}{2}(|b^i(t)| - sgn(b^i(t)))\sigma_{b^i}(t)^2\right] dt + \frac{1}{2}\sum_{i=1}^{N} sgn(b^i(t))\sigma_{b^i}(t)dW(t)$$

Which implies that total debt follows the following stochastic process

$$d\Gamma(t) = \mu_\Gamma(t)dt + \sigma_\Gamma(t)dW(t)$$

where

$$\begin{aligned}
\mu_\Gamma(t) &= \frac{1}{2}\sum_{i=1}^{N}\left[sgn(b^i(t))\left(\mu_{b^i}(t) - \frac{1}{2}\sigma_{b^i}(t)^2\right) + \frac{1}{2}|b^i(t)|\sigma_{b^i}(t)\right] \\
\sigma_\Gamma(t) &= \frac{1}{2}\sum_{i=1}^{N} sgn(b^i(t))\sigma_{b^i}(t)
\end{aligned}$$

Next, define total financial leverage to be $L(t) = \frac{\Gamma(t)}{S(t)}$ and recall Itô's quotient rule: $d\frac{X}{Y} = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2\right)$. So, we can define the dynamics of total financial leverage:

$$\begin{aligned}
dL(t) = d\frac{\Gamma(t)}{S(t)} &= \frac{\Gamma(t)}{S(t)}\left(\frac{d\Gamma(t)}{\Gamma(t)} - \frac{dS(t)}{S(t)} - \frac{d\Gamma(t)}{\Gamma(t)}\frac{dS(t)}{S(t)} + \left(\frac{dS(t)}{S(t)}\right)^2\right) \\
&= \frac{\Gamma(t)}{S(t)}\left[\left(\frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{\Gamma(t)} - \mu_S(t) + \sigma_S(t)^2\right)dt + \left(\frac{\sigma_\Gamma(t)}{\Gamma(t)} - \sigma_S(t)\right)dW(t)\right]
\end{aligned}$$

Thus, total financial leverage follows a geometric Brownian motion such that

$$\frac{dL(t)}{L(t)} = \mu_L(t)dt + \sigma_L(t)dW(t)$$

where

$$\begin{aligned}
\mu_L(t) &= \frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{\Gamma(t)} - \mu_S(t) + \sigma_S(t)^2 \\
\sigma_L(t) &= \frac{\sigma_\Gamma(t)}{\Gamma(t)} - \sigma_S(t)
\end{aligned}$$

□

Appendix C. Numerical Simulation Method

Gathering all of the stochastic processes we have the following definitions to describe the evolution of the economy:

$$
\begin{aligned}
\frac{dD(t)}{D(t)} &= \mu_D dt + \sigma_D dW(t) \\
\frac{d\omega^i(s)}{\omega^i(s)} &= \mu_{\omega^i}(s)dt + \sigma_{\omega^i}(s)dW(t) \\
\frac{dH_0(t)}{H_0(t)} &= -r(t)dt - \theta(t)dW(t) \\
\theta(t) &= \frac{\sigma_D}{\xi(t)} \\
r(t) &= \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t)+\phi(t)}{\xi(t)^3}\sigma_D^2
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\mu_{\omega^i}(t) &= (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \theta(t)^2\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right] \\
\sigma_{\omega^i}(t) &= \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) \\
\xi(t) &= \sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i} \\
\phi(t) &= \sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i^2}
\end{aligned}
$$

given a set of initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$. All of the above variables can be determined as a function of the realization of the risk process $W(t)$. If we combine those values with an estimation of asset prices and the following formulas

$$
\begin{aligned}
\theta(t) &= \frac{\mu_s + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)} \\
\frac{dS(t)}{S(t)} &= \mu_s(t)dt + \sigma_s(t)dW(t)
\end{aligned}
$$

we can back out the coefficients $\mu_S(t)$ and $\sigma_S(t)$ and study the dynamics of the economy, as well as characteristics of asset prices.

Note that it can be shown (Oksendal (1992)) that if a stochastic process $Z(t)$ follows a geometric Brownian motion with drift and diffusion $\mu_Z(t)$ and $\sigma_Z(t)$, then

$$
Z(t+\Delta t) = Z(t)e^{(\mu_Z - \frac{1}{2}\sigma_Z^2)\Delta t + \sigma_Z(W(t+\Delta t)-W(t))} \tag{24}
$$

The numerical scheme follows the following steps:

1. Specify a time discretization such that $t \in \{0, 1, ..., T\}$ and a time step Δt. Note that the specification of parameters and this time step will determine the discretization as being yearly, quarterly, monthly, etc.
2. Specify a set of agents indexed by $i \in \{1, ..., N\}$ for some number N and each agent's risk aversion parameter γ_i.
3. Specify initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$.
4. Simulate a process $\{dW(t)\}_{t=0}^T$ where $dW(t) \sim \mathcal{N}(0, \Delta t)$.
5. Using (23), (24), and the simulated Wiener process, for each period $t \in \{0, 1, ..., T\}$ calculate $\{D(t), \{\omega^i(t)\}_{i=1}^N, r(t), \theta(t), \xi(t), \phi(t)\}$.
6. Using the monte-carlo approach described in appendix C.1, for each period $t \in \{0, 1, ..., T\}$ calculate $\hat{S}(t)$, $\hat{\sigma_S}(t)$, and $\hat{\mu_S}(t)$.
7. Given the process for $\hat{S}$, calculate wealth $X^i(t)$ for each period using the definitions $X^i(0) = \omega^i(0)S(0) + b^i(0) = \omega^i(0)S(0)$ (where $b^i(t)$ is risk free bond holdings and we assume agents enter the model with no savings/debt) and (3).
8. Calculate any measures you might find enlightening!

C.1. Estimating Asset Prices

The expression we wish to estimate is given by

$$S(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u) du$$

In order for the integral to be defined, it must be that the integrand converges towards zero as $u \to \infty$. If this is the case, then we could estimate the integral by truncating the upper bound at some level, $t + T$. In this way we would look to approximate the true asset price in the economy by another:

$$S(t) \approx S^*(t) = \mathbb{E}_t \int_t^{t+T} \frac{H_0(u)}{H_0(t)} D(u) du$$

This expression can easily be estimated by monte-carlo. To estimate the integral, I'll use the trapezoid rule, but note that in the numerical simulation this is exact, given that time is discretized. Define the discretization by partitioning the interval $(t, t+T)$ into H evenly spaced intervals such that Δ_t is the distance between points in the partition. To estimate the expectation, I will use monte-carlo (see Casella and Robert (2013)) by sampling M paths for the process $W(t)$ and simulating the economy along these paths to extract processes H_0 and D. Indicating draws by a super-script m the estimator is given by:

$$\hat{S}^{*}(t) = \frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{2}\left(D(t) + \frac{H_0^m(T)}{H_0(t)}D^m(T)\right)\right.$$
$$\left. + \sum_{i=1}^{M-1}\frac{H_0^m(t+\Delta_t i)}{H_0(t)}D^m(t+\Delta_t i)\right] \tag{25}$$

Given the computational simplicity of this expression, it can be calculated quite easily using parallel methods on a graphics processing unit (GPU). I use an element-wise product and pairwise summation to calculate the expectation, resulting in a 300 times speed up (code available upon request).

To estimate, the steps are as follows for a given distribution of $\{\gamma_i\}_{i=1}^N$ and an initial condition for the distribution of wealth $\{\omega^i(t)\}_{i=1}^N$:

1. Simulate M sample paths for $dW(t)$ of length T, where M is an integers, using the knowledge that $dW(t) \sim \mathcal{N}(0,t)$.
2. Using the M sample paths, simulate the evolution of M different economies populated by the same agents under the same initial condition. Extract the values for $(H_0(t), D(t))$.
3. Calculate asset prices at period t as the Monte carlo approximation given in (25).

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